The Billing-Mahler Theorem

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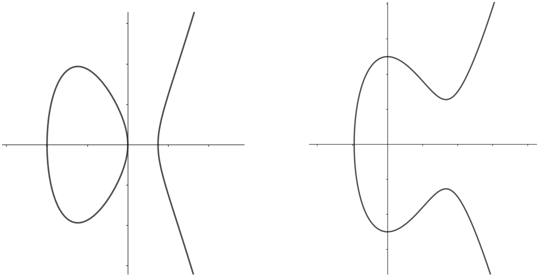
July 9, 2024

Elliptic Curve

An elliptic curve over $\mathbb{Q}$ is a non-singular curve given by

$$E = \left\{(x, y) \mid y^2 = f(x) = x^3 + ax^2 + bx + c\right\} \cup \{\mathcal{O}\},$$

where a, b, c are integers.



We want to find the rational points on curves like these.

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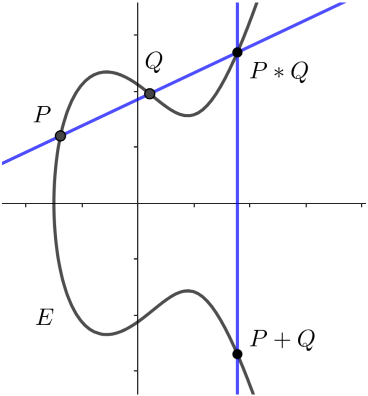

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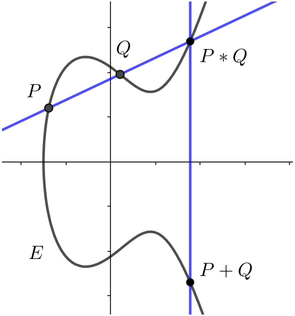
- Closure:
- Identity:
- Inverses:
- Associativity:



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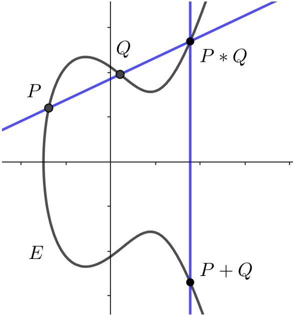

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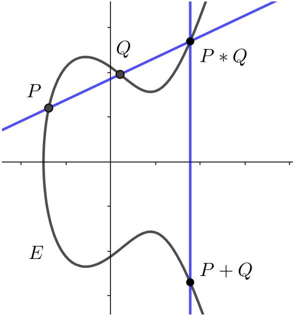

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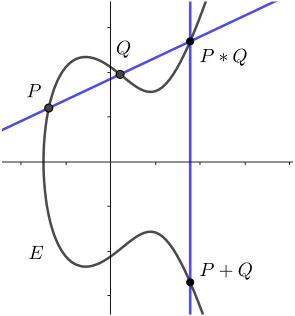

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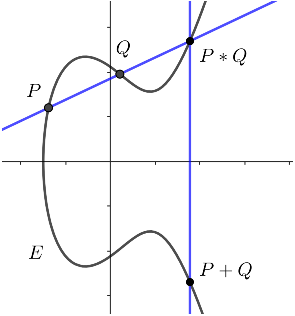
- Identity: Is the point $\mathcal{O}$.
- Inverses: Reflection over x-axis.
- Associativity: Can be checked using explicit formulas for the $+$ operation. □



The Group Law

One way that the group law is useful, is that given an initial set of rational points, we can obtain more rational points using the group law.

Example

Let E be the elliptic curve given by

$$y^2 = f(x) = x^3 - 4x + 4.$$

Plugging in $x = 0$ and $x = 1$ gives solutions $(0, \pm 2)$ and $(1, \pm 1)$.

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The line through A and B is $y = 3x - 2$, so we plug back into $y^2 = f(x)$ to get $0 = x^3 - 9x^2 + 8x$. Then we factor to find that $x = 0$, $x = 1$, or $x = 8$.

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Plugging $x = 8$ into $y^2 = f(x)$, we get $A * B = (8, 22)$, and $A + B = (8, -22)$.

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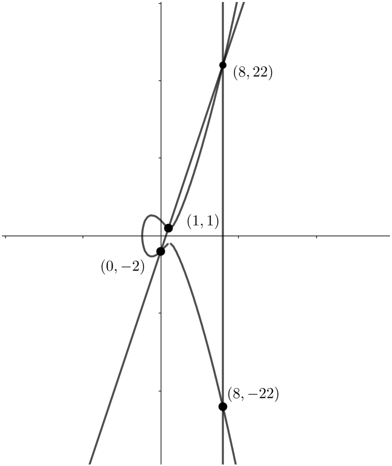

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So, for any elliptic curve E (defined over $\mathbb{Q}$), there is a finite set of rational points such that we can find all other rational points on E by adding points from the starting set in various ways.

Points of Finite Order

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Definition

A point of finite order is a point P such that

$$\underbrace{P + P + \cdots + P}_{n \text{ times}} = nP = \mathcal{O}$$

for some non-negative integer n. We may also say that P is a *torsion point*.

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We can check that this is $(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$, where x_1, x_2, x_3 are the roots of the cubic, so the discriminant is nonzero if and only if the roots are all distinct.

Points of Finite Order

Theorem (Nagell-Lutz)

If $P = (x, y)$ is a rational torsion point (rational point of finite order) on elliptic curve E given by

$$y^2 = f(x) = x^3 + ax^2 + bx + c,$$

then:

- *x and y are both integers*
- *either $y = 0$ or y divides the discriminant of f.*

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So, through the Nagell-Lutz theorem we can obtain a complete list of the rational points of finite order in a finite number of steps.

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For $y^2 = 1$, any potential rational solutions will have an x-coordinate satisfying $0 = x^3 - 4x + 2$. The rational roots of $x^3 - 4x + 2$ must divide 2, so they would be in $\{\pm 1, \pm 2\}$. Checking each possibility $\implies$ no rational roots in this case.

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So, the only possibility is when $y = 0$. Rational roots to $0 = x^3 - 4x + 3$ would divide 3, and checking each possibility, we get $(1, 0)$ as the only rational torsion point on E.

Possible Orders

It turns out that we can write down elliptic curves with rational points that have order 2, 3, 4, 5, 6, 7, 8, 9, 10, and 12, but never an elliptic curve with a rational point of order 11. This is what is known as the Billing-Mahler theorem (proven in 1940):

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Theorem (Mazur)

If $E(\mathbb{Q})$ has a point of finite order m, then either $1 \leq m \leq 10$ or $m = 12$.