Hadwiger-Nelson Problem

Cecilia Belej

July 14, 2024

Table of Contents

1. The Hadwiger-Nelson Problem
2. Lower Bound
3. Upper Bound
4. Attempts to Narrow the Upper Bound
5. Other Variants

Problem (1950)

What is the minimum number of colors required to color a plane so that no two points at unit distance from each other are the same color? (Open Problem)

Definitions

Vertex: a point on the plane
Edge: a segment connecting two vertices (we may also say that these two vertices are adjacent, or that they are the endpoints of an edge)
Chromatic number (denoted by $\chi(\mathbb{E}^2)$): the minimum number of colors required for a graph so that no two adjacent points have the same color
De Brujin-Erdős Theorem: If all finite subgraphs of a complete graph can be colored with c colors, then the same chromatic number applies to the whole graph... basically this allows us to study finite graphs instead of the infinite plane

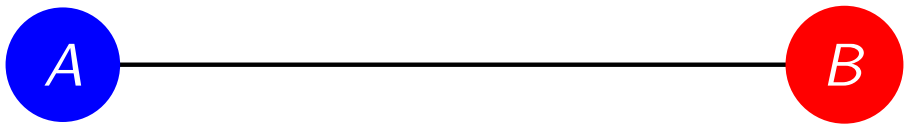

Figure 1: Edge with 2 vertices (Chromatic number 2)

Lower Bound

Discovered in 1960 by brothers William and Leo Moser
Chromatic number: 4
Vertices: 7

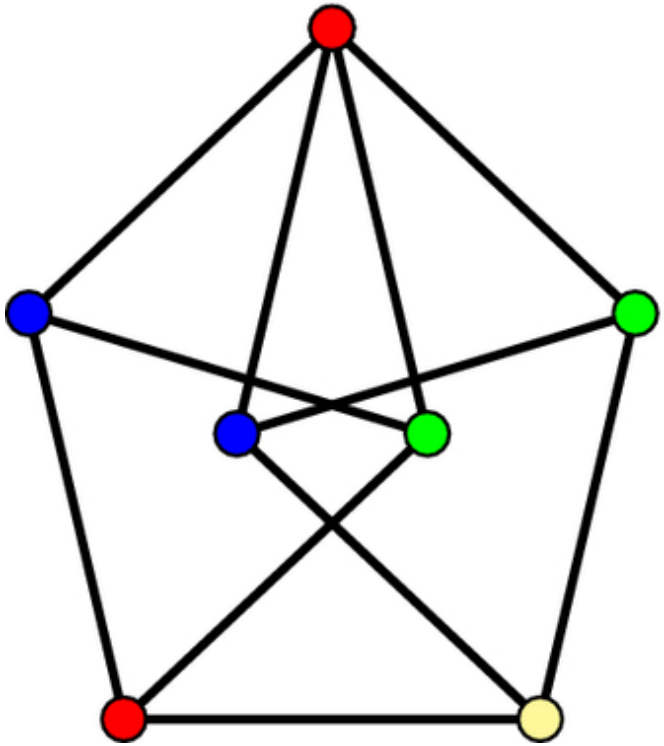

Figure 2: Moser Spindle

Lower Bound (Continued)

Discovered in 2018 by Aubrey de Grey
Chromatic number: 5
Vertices: 1581

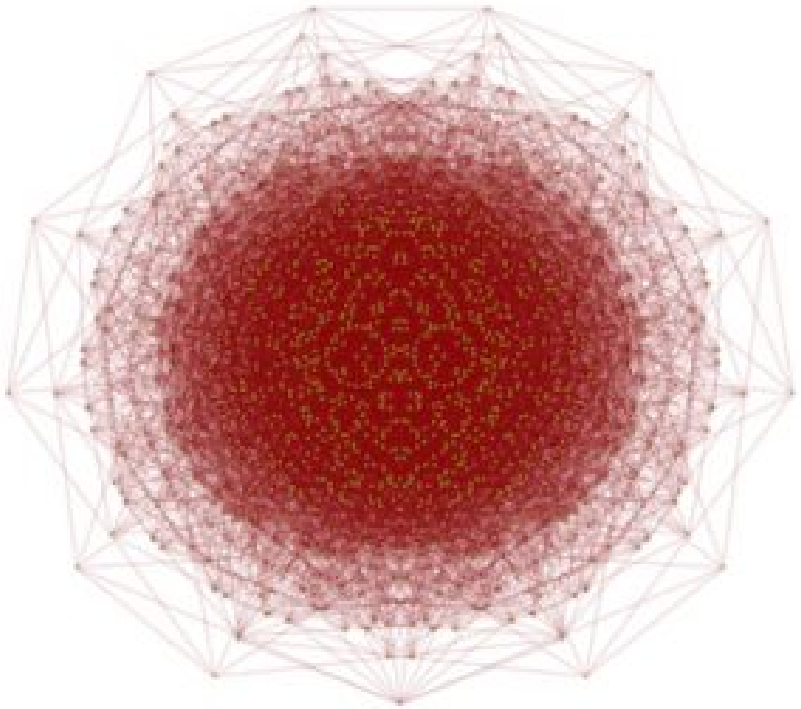

Figure 3: De Grey's graph

Upper Bound- Hexagons

1. Surround one regular hexagon by six others to make a sort of flower shape (7 different colors)
2. Tessellate this flower shape to cover the plane
3. Lay any unit-distance graph over the hexagons; the color of the hexagon where each vertex lies determines its color

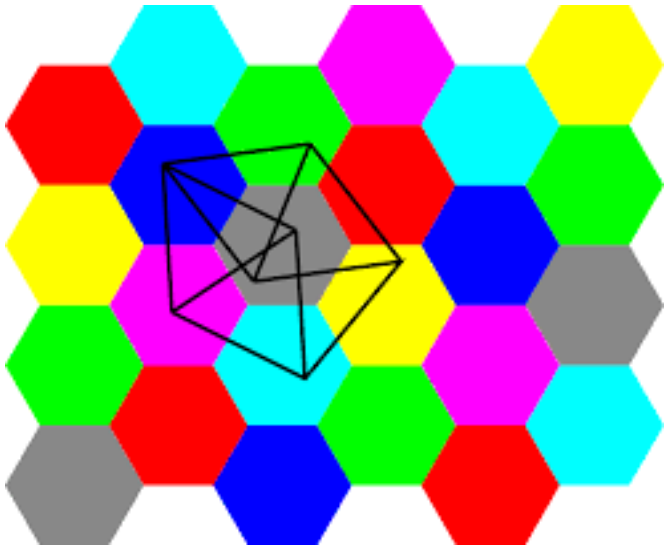

Figure 4: Moser Spindle laid over hexagons

Hexagon Size

For this to work, the hexagons need to be a certain size in relation to the unit distance. To prevent any two endpoints of the same edge from being the same color, the hexagons need to be small enough so that an entire edge can't fit in one hexagon, and they need to be large enough to prevent a single edge from being able to bridge the distance between two hexagons of the same color.

Solving for x

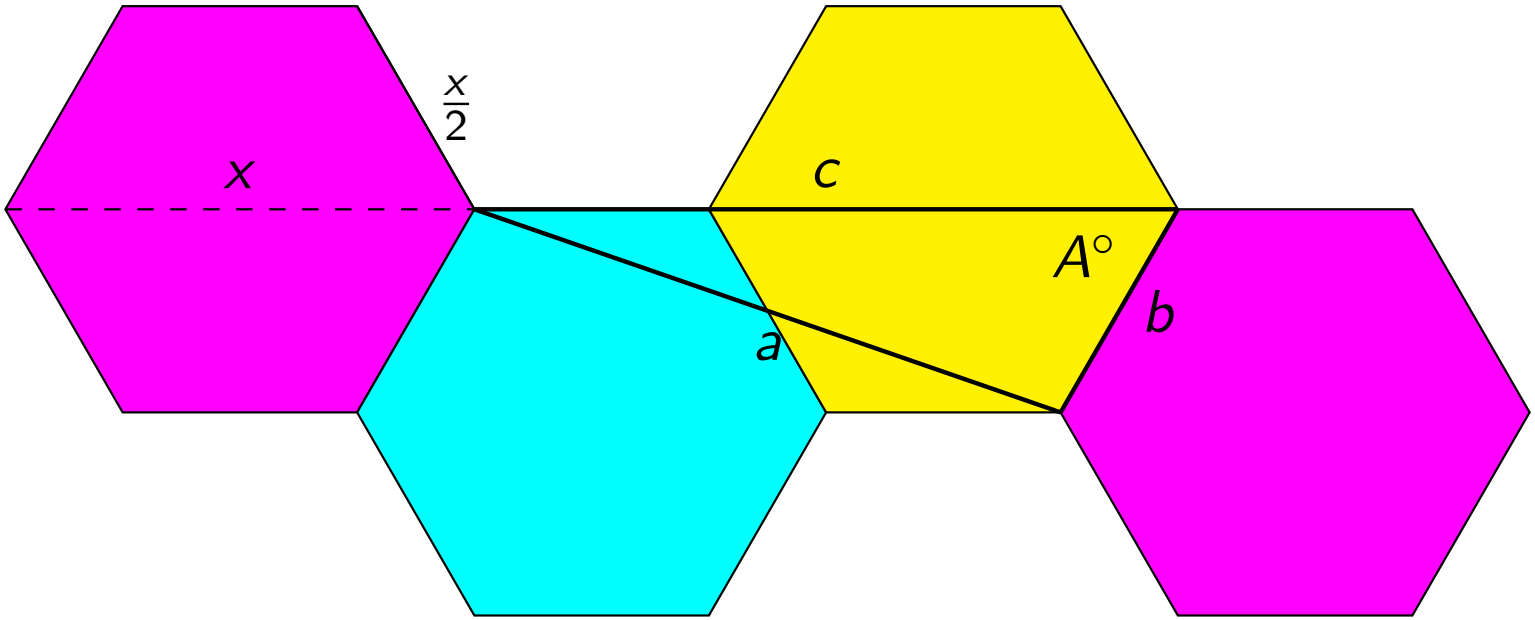

Figure 5: Shortest distance between hexagons of the same color: a

$c = 1.5x$
$b = 0.5x$
$A = 60°$
$a = \frac{\sqrt{7}x}{2}$ (Law of Cosines)

Conclusion

Therefore, the length l of each edge in the graph must lie in the range $x < l < \frac{\sqrt{7}x}{2}$, so when we set l to 1, the diameter x of each hexagon must lie in the range $\frac{2}{\sqrt{7}} < x < 1$.
Therefore, no more than 7 colors will be needed to color a unit-distance graph. Now, we can conclude the following:
The chromatic number of the plane $\chi(\mathbb{E}^2)$ lies in the range $5 \leq \chi(\mathbb{E}^2) \leq 7$.

Other Shapes

At this point, you may be wondering if other shapes could work, resulting in a smaller chromatic number.

The only regular polygons that tessellate are hexagons, squares, and triangles, so we can try tessellating squares and triangles to see why these don't produce a narrower upper bound for the chromatic number.

Squares

1. Take one square and surround it by four others (five different colors)
2. Tessellate these cross shapes to cover the plane (shown on next slide)
3. If the side length is s, then the diagonal of the square is $\sqrt{2}s$
4. The greatest distance within a square is $\sqrt{2}s$, but the shortest distance between two squares of the same color is s

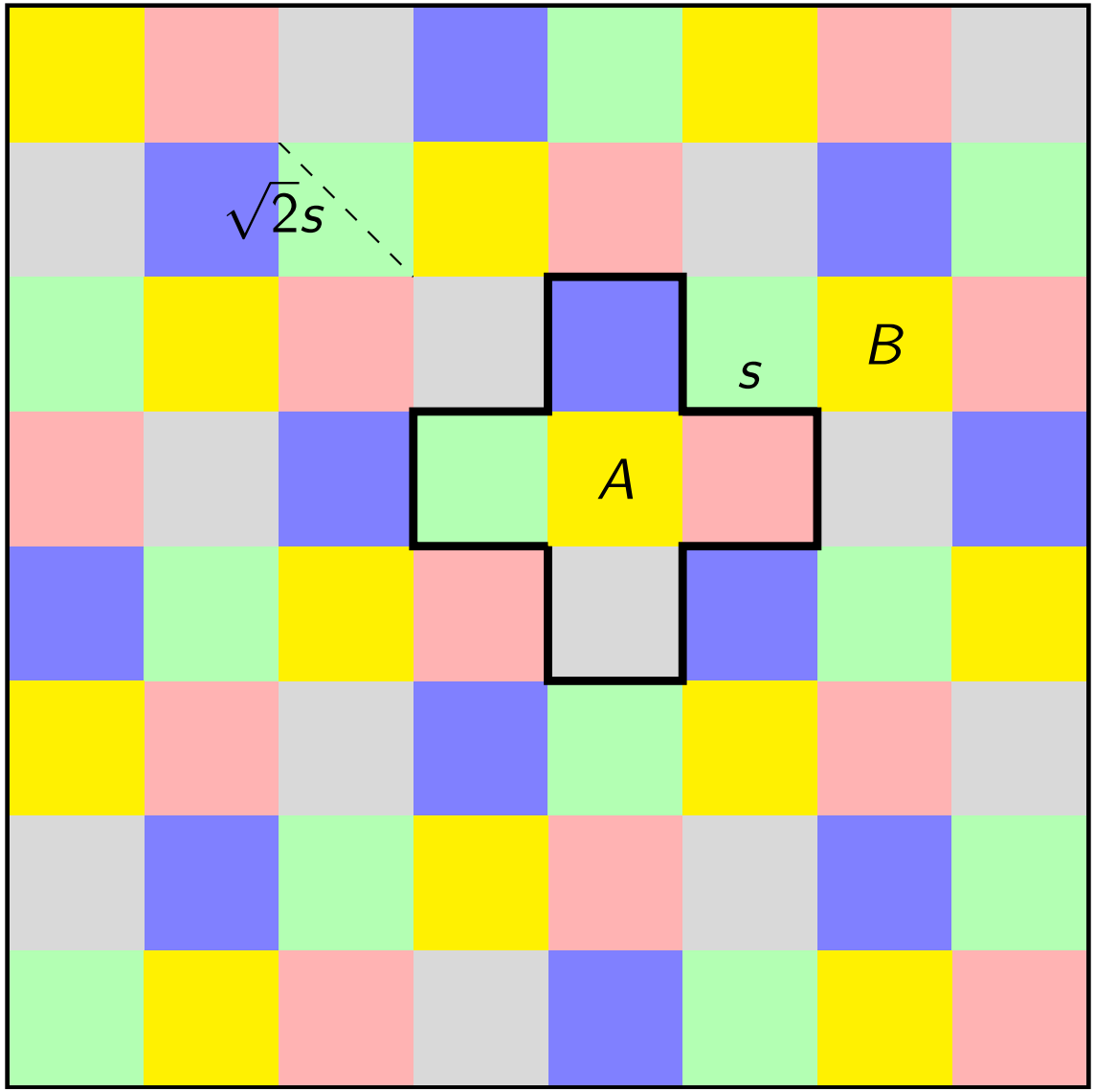

Figure 6: Tessellation of squares

Triangles

Triangle Tessellation

1. Take six equilateral triangles, each of a different color, and fit them together to make a hexagon
2. Tessellate these hexagons to cover the plane (shown on next slide)
3. If the altitude is a, then the side length is $\frac{2a\sqrt{3}}{3}$, which is the longest distance within a single triangle
4. For each triangle in this tessellation, six triangles lie a units away

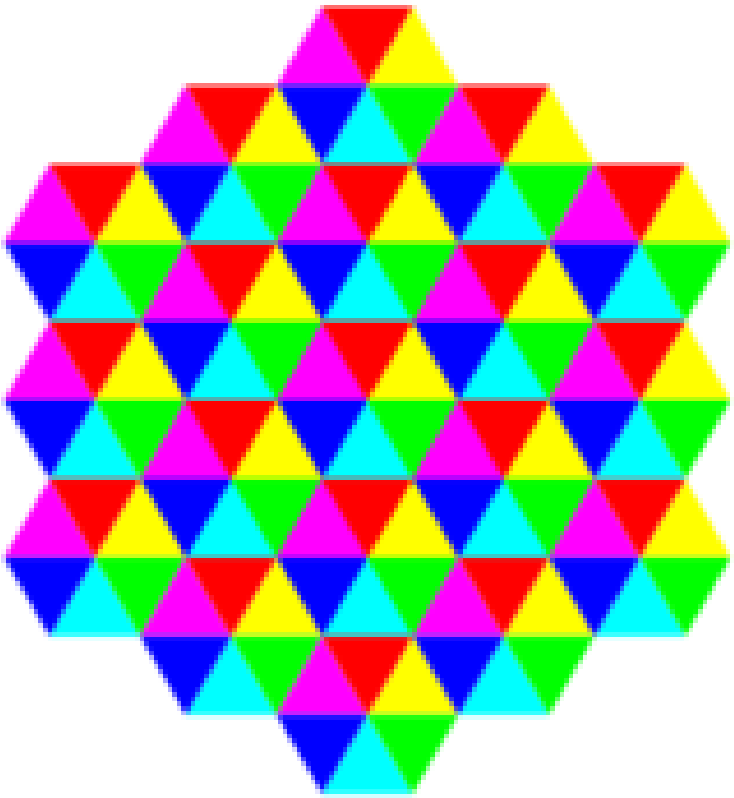

Figure 7: Triangle Tessellation

Variants

What is the chromatic number of the plane if vertices can be distance 1 or distance d apart?

What is the chromatic number of the rational graph?

The End

Thanks for watching!