BOUNDS AND VARIANTS OF THE HADWIGER-NELSON PROBLEM

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1. Abstract

The Hadwiger-Nelson Problem, first proposed by Edward Nelson in 1950, is an open problem in mathematics regarding graph coloring. Although the answer is unknown, it has been narrowed down to three possible solutions. This paper will discuss the upper and lower bounds for the answer, and it will also discuss variants to the problem.

2. Introduction

Question 2.1. *What is the minimum number of colors required to color a plane so that no two points at unit distance from each other are the same color?*

Currently, it is known that $5 \leq \chi \leq 7$ for the Hadwiger-Nelson problem. In this paper, both of these bounds will be explained. Then, the paper will explore the popular variant to this problem that forbids a second distance between same-colored points. Lastly, we will look at another variant to this problem that only deals with the rational plane, rather than considering all real numbers.

3. Background

In the Hadwiger-Nelson Problem, it wouldn't be possible to study all points in the Euclidean plane $\mathbb{E}^2$; rather, mathematicians focus on certain sections of it.

Definition 3.1 (Axiom of Choice)**.** The Axiom of Choice in set theory states that for any collection of nonempty sets, a choice function can choose one element from each set, creating a new set.

Similar to the Axiom of Choice is the de Bruijn-Erdős Theorem. This theorem essentially makes the same claim but is more specific to graph theory, and it clarifies the solving of this problem.

Definition 3.2 (de Bruijn-Erdős Theorem)**.** This states that for an infinite plane, the chromatic number χ of all the finite sub-graphs will be at most the chromatic number of the infinite graph [4].

Therefore, solving this problem doesn't actually have to involve an infinite plane; rather, finite graphs can be analyzed. Since they are elements of the set that is the infinite plane, what is true for the finite graphs also applies to the infinite graph expressed in this problem.

Before proceeding with the proofs of the possible solutions, we will define some terminology used in this paper:

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Definition 3.3 (Vertex)**.** A vertex, in the context of graph theory, is a point on the plane. In the Hadwiger-Nelson problem, we look at the vertices included in the graphs we've chosen to focus on (see de Bruijn-Erdős Theorem).

Definition 3.4 (Edge)**.** An edge, in the context of graph theory, is a segment connecting two vertices. We may also say that these two vertices are adjacent, or that they are the endpoints of an edge.

Definition 3.5 (Chromatic number)**.** The chromatic number of the Euclidean plane, denoted by $\chi(\mathbb{E}^2)$, is the minimum number of colors required for a graph so that no two adjacent points (separated by distance one) have the same color.

4. Lower Bound

In 1961, the Moser Spindle (Figure 1) was discovered by the brothers William and Leo Moser. This simple graph, consisting of only 7 vertices, requires 4 colors, which proved that χ was bounded below by 4. For a long time after that, no significant progress was made on the problem.

Then, in 2018, amateur mathematician Aubrey de Grey constructed a unit-distance graph with chromatic number 5 [2]. Shown below in Figure 2, this graph consists of 1581 vertices. Clearly, it is much more complicated than the Moser Spindle and many other graphs with chromatic number 4; however, de Grey's original graph has been reduced many times to form smaller graphs that also have chromatic number 5. Mathematicians believe that with these smaller graphs, they can better understand how to construct graphs with a high chromatic number, and it may lead to more progress on the problem. After all, de Grey used the Moser Spindle in the construction of his graph.

5. Upper Bound

Theorem 5.1. *The upper bound of the chromatic number is $\chi \leq 7$*

Proof. The upper bound of χ can be proved by tessellating regular hexagons. We can surround one hexagon by six others to make a sort of flower shape, with each of the seven hexagons being a different color. This flower shape can be tessellated to cover the plane. Any unit-distance graph can be laid over the hexagons, and the color of the hexagon where each vertex lies determines its color. For example, Figure 3 shows the Moser Spindle laid over this tessellation of hexagons. Each vertex would be colored the same as the hexagon on which it lies. □

For this to work, the hexagons need to be a certain size in relation to the unit distance. To prevent any two endpoints of the same edge from being the same color, the hexagons need to be small enough so that an entire edge can't fit in one hexagon, and they need to be large enough to prevent a single edge from being able to bridge the distance between two hexagons of the same color.

We can focus on smaller sections of this tessellation to determine the shortest distance between two hexagons of the same color, as shown in Figure 4. If we set x equal to the diameter of the hexagon, then the length of each side is $\frac{x}{2}$. The length of a would be the shortest possible distance between two hexagons of the same color (in this case, two of the pink hexagons); b has a length of $\frac{x}{2}$, c has a

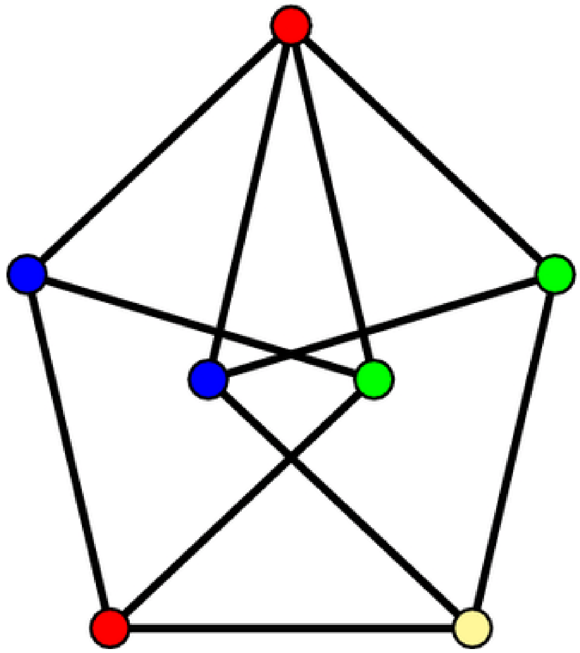

FIGURE 1. Moser Spindle

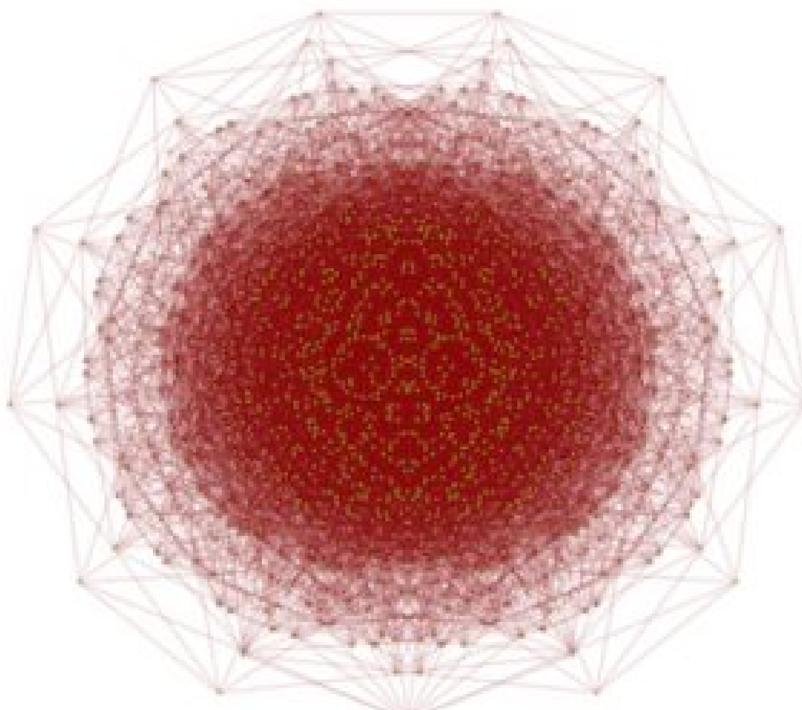

FIGURE 2. de Grey's graph

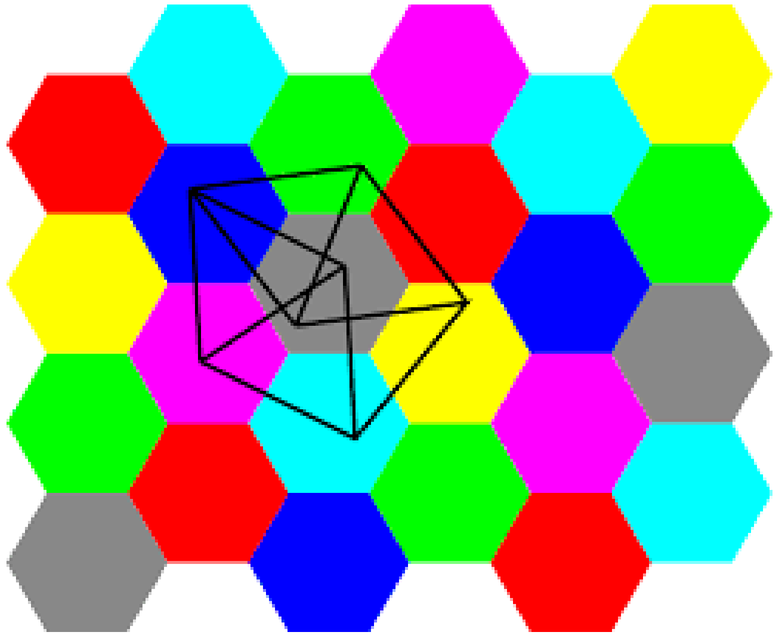

FIGURE 3. Moser Spindle laid over hexagons

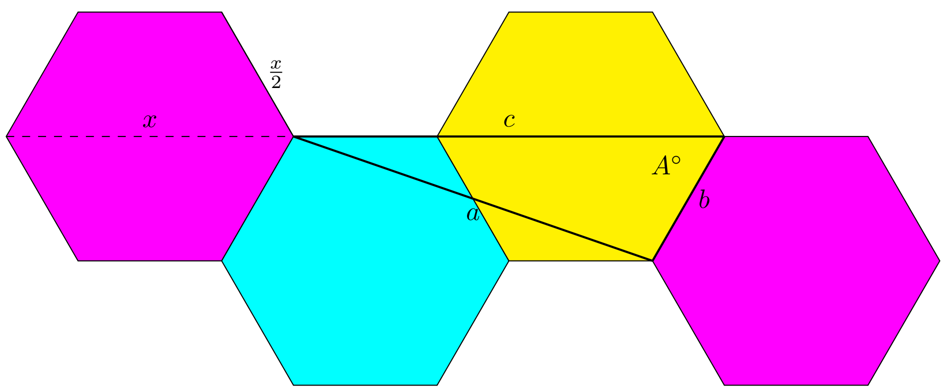

FIGURE 4. Distance between hexagons of the same color

length of $1.5x$; and angle A has a measure of $60°$ (since c bisects an interior angle of a regular hexagon, which would have a measure of $120°$).

To solve for a, we can use the Law of Cosines, as shown below:

$$a^2 = (1.5x)^2 + (\frac{x}{2})^2 - 2(1.5x)(\frac{x}{2})(cos60)$$
$$a^2 = \frac{7x^2}{4}$$
$$a = \frac{\sqrt{7}x}{2}$$

Therefore, the length l of each edge in the graph must lie in the range $x < l < \frac{\sqrt{7}x}{2}$, so when we set l to 1, the diameter x of each hexagon must lie in the range $\frac{2}{\sqrt{7}} < x < 1$. This ensures that the vertices of each edge will lie in hexagons of different colors. Since 7 colors are used for this proof, no more than 7 colors will be needed to color a unit-distance graph; that is, $\chi \leq 7$. Now, based on the proofs in this section and the previous section, we can conclude the following:

Theorem 5.2. *The chromatic number of the plane $\chi(\mathbb{E}^2)$ lies in the range $5 \leq \chi(\mathbb{E}^2) \leq 7$.*

6. Attempts to Narrow the Upper Bound

Seeing how tessellating hexagons sets an upper bound for the chromatic number of the graph, we may wonder if it would be possible to tessellate polygons with fewer sides in order to use fewer colors. Aside from hexagons, the only regular polygons that tessellate are squares and triangles (for example, if we surrounded one regular pentagon by five others, there would be extra space between the pentagons).

We can take one square and surround it by four others to make a sort of cross shape, with each of the five squares being a different color. These cross shapes can be tessellated, as shown in Figure 5. Now, just like we did for the hexagons, we must determine acceptable dimensions of the squares such that one edge cannot have both vertices in squares of the same color. For this to work here, the unit distance must be longer than the diagonal of the squares and shorter than the shortest distance between two squares of the same color.

If we set the side length of each square equal to s, then the length of the diagonal will be $\sqrt{2}s$. Now, looking back at Figure 5, we can see that the yellow squares labeled A and B are separated by a length of s. Of course, we can't have a length that would be greater than $\sqrt{2}s$ but less than s, so a tessellation of squares cannot narrow the upper bound of χ.

Now, let's examine a tessellation of triangles. We can take six equilateral triangles, each of a different color, and fit them together in a hexagon shape; then, these hexagons can be tessellated, as shown in Figure 6. If we set the length of each triangle's altitude to a, then their side lengths are equal to $\frac{2a\sqrt{3}}{3}$, which is the longest distance within a single triangle. Now, consider any triangle in this tessellation. Six triangles of the same color lie a units away, which is a shorter distance than the side length, $\frac{2a\sqrt{3}}{3}$.

Therefore, the unit distance d of the graph would need to lie in the range $\frac{2a\sqrt{3}}{3} < d < a$; however, no measurement of the altitude a exists for this inequality. Here, we have the same problem as we did for the tessellation of squares: no possible scaling of the triangles would guarantee that adjacent vertices in a unit-distance graph would lie in triangles of different colors.

We can see that more circular shapes generally work better for this. For the longest distance within a shape, we want that distance to occur in as many directions as possible to prevent the distance from being long enough to reach the next shape of the same color. Circles, of course, cannot be tessellated alone; they would leave smaller spaces between them, and it still would not be possible to find a length greater than the circles' diameter but less than the distance between shapes of the same color. Regular hexagons work nicely because of their ability to tessellate without any other shapes involved, and them having a more circle-like shape than

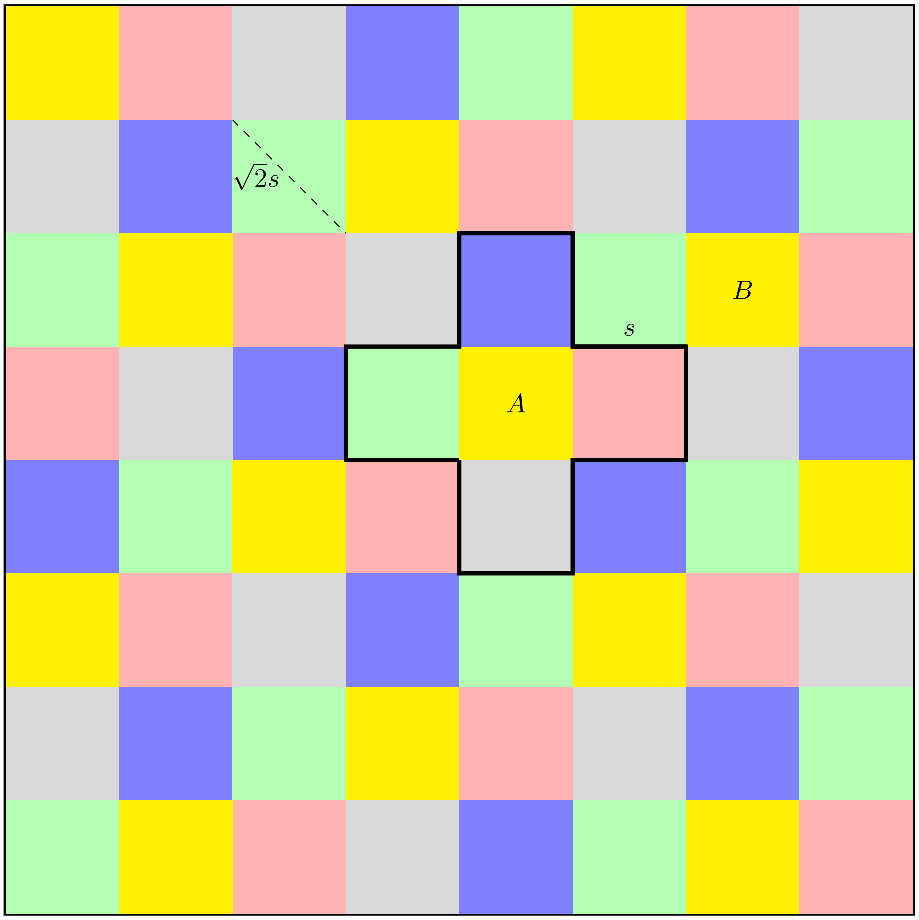

Figure 5. Tessellation of squares

squares and triangles makes it possible to have an edge longer than their diameter and shorter than the distance between same-colored hexagons.

Since tessellations of neither squares nor triangles prove a lower value of χ, the upper bound remains at seven.

7. Allowing A Second Distance

The original problem only looks at edges with a length of one, but certain variants of this problem allow edges of two different distances: 1 and d for some $d \neq 1$ (not to be confused with d from earlier in this paper that denoted unit distance). For example, instead of edges only connecting points at unit distance from each other, a graph may consist of edges connecting points at a distance of both one and two. The chromatic number of a graph that only allows edges of unit distance is denoted by $\chi(\texttt{\{1\}})$, whereas the chromatic number of a graph with edges of distance 1 or d is denoted by $\chi(\texttt{\{1,d\}})$.

One thing to note here that applies to all values of d is that any lower bound discovered for the original problem is also true for this variant, since including

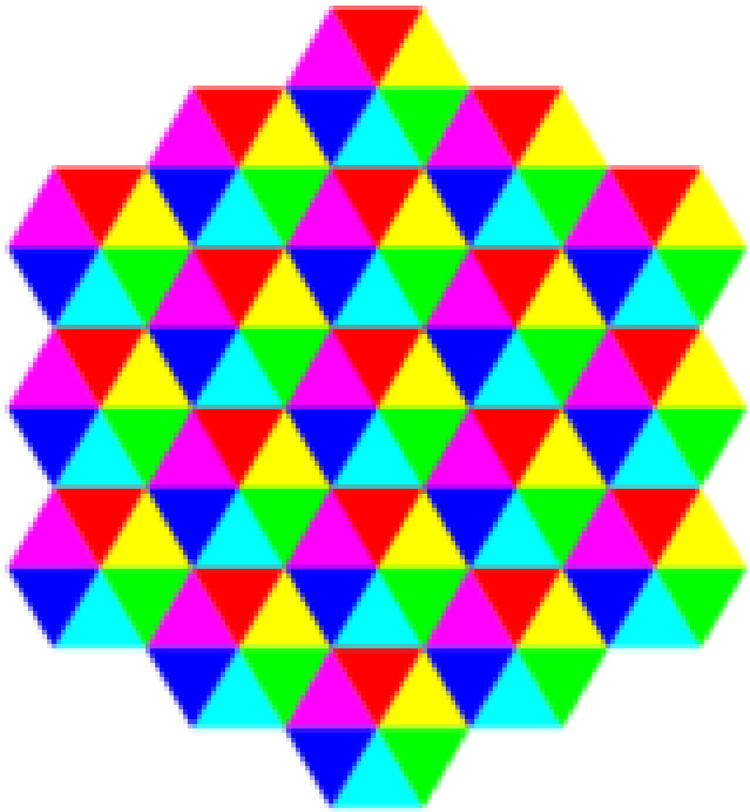

FIGURE 6. Tessellation of triangles

another possible measurement of edges only adds opportunities for more points in a graph without eliminating any. The lower bound of 5 for χ was proven by de Grey's graph, where only a distance of one is permitted; therefore, regardless of the value of this other distance d, it would only add more edges to the graph. Because of this, de Grey's graph would still have a chromatic number of at least five. If someone were to create a unit-distance graph with chromatic number six in the future, then the lower bound of χ for graphs with both distances 1 and d would also be six. Basically, $\chi(\texttt{\{1,d\}})$ can never be less than $\chi(\texttt{\{1\}})$.

However, graphs that use less vertices are generally simpler to understand and easier to work with. Although large, complicated graphs can sufficiently prove a lower bound for the chromatic number, simpler graphs are easier to study and may contribute to future progress- similar to how mathematicians have reduced de Grey's 5-chromatic 1581-vertex graph. Even though de Grey's proof that $\chi(\texttt{\{1\}}) \geq 5$ also proves that $\chi(\texttt{\{1,d\}}) \geq 5$, we can take advantage of having the second distance d to construct much simpler graphs of chromatic distance 5. Geoffrey Exoo's and Dan Ismailescu's paper "The Hadwiger-Nelson Problem with Two Forbidden Distances" [3] provides many constructions of 5-chromatic graphs that are far simpler than any known graph proving $\chi(\texttt{\{1\}}) \geq 5$.

7.1. **Revisiting the Hexagons.** The hexagon tessellation from earlier in this paper can be used to provide an upper bound for certain values of d in this variant. We proved that $x < l < \frac{\sqrt{7}x}{2}$ for the diameter x of each hexagon and the length of each edge l. We used this in the proof of the upper bound of χ for unit-distance graphs, but the range of values provides some flexibility that allows us to set the same upper bound for certain values of d. We can find values of d for which the upper bound $\chi \leq 7$ exists by determining when this inequality is true for both $l = 1$ and $l = d$.

For the inequality $x < l < \frac{\sqrt{7}x}{2}$, both 1 and d must lie within that range. We can set 1 equal to either the lower bound of this range in order to find the upper bound of d, or vice versa. When we plug in 1 for either x or $\frac{\sqrt{7}x}{2}$, we end up with the following:

$$1 < l < \frac{\sqrt{7}}{2}$$
$$OR$$
$$\frac{2}{\sqrt{7}} < l < 1$$

The diameter of the hexagons can be adjusted depending on the value of d and which of the above inequalities we use. Therefore, $\chi(\texttt{\{1,d\}}) \leq 7$ when $\frac{2}{\sqrt{7}} < d < \frac{\sqrt{7}}{2}$.

This section presented a variant to the Hadwiger-Nelson problem that uses two distances instead of one; however, infinitely many values of d in this subsection can be used in the same graph with its chromatic number still being bounded above at seven. As long as all the edges in a graph have a length within the range $x < l < \frac{\sqrt{7}x}{2}$, then $\chi \leq 7$.

7.2. **Solving for χ when $d = \frac{1+\sqrt{5}}{2}$.** The lower bound for $\chi(\texttt{\{1,d\}})$ when $d = \frac{1+\sqrt{5}}{2}$ is 5; however, this can be proven by a much simpler graph than those used to prove $\chi(\texttt{\{1\}}) \geq 5$.

If we consider a regular pentagon with side lengths of 1, then the diagonals of the pentagon have a length of $\frac{1+\sqrt{5}}{2}$. Figure 8 shows this pentagon with all its diagonals, so it is a graph with distances 1 and d when $d = \frac{1+\sqrt{5}}{2}$ [5]. Using Proof by Contradiction, we can prove that if $d = \frac{1+\sqrt{5}}{2}$, then $\chi(\texttt{\{1,d\}}) \geq 5$.

Proof. If the chromatic number of the graph in Figure 8 was no greater than 4, then we could color the vertices using only 4 colors. As shown in the figure, we can color vertices A, B, C, and D using blue, red, violet, and yellow respectively. Since any two of these points are connected in some way, no two of them may be the same color. However, none of these colors can be reused to color vertex E; it is connected to all four of the other vertices, either by one of the sides of the pentagon (vertices A and D) or by one of its diagonals (vertices B and C). Therefore, since

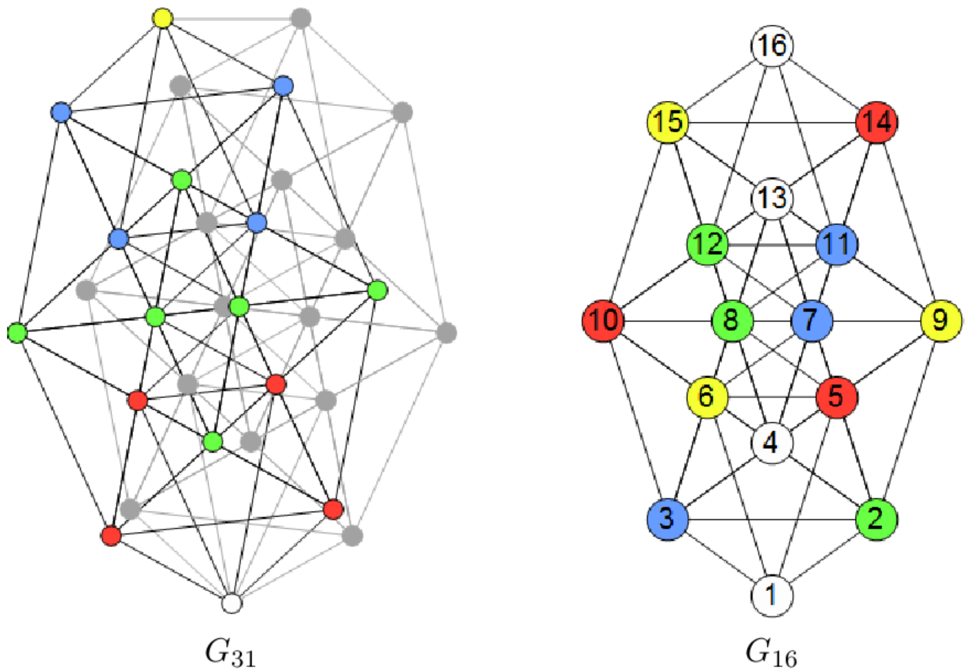

FIGURE 7. A 6-chromatic graph with 2 distances

the graph cannot be colored using only four colors, the chromatic number must be at least five. □

Later, the proof for $\chi(\texttt{\{1,d\}})$ for $d = \frac{1+\sqrt{5}}{2}$ was improved by a graph with a chromatic number of 6 [6]. The graph was made using the same idea that the diagonal of a regular pentagon with side length 1 is $\frac{1+\sqrt{5}}{2}$. Figure 7, included the cited paper, shows this graph.

8. Chromatic Number of the Rational Plane

Up until now, we've dealt with the real plane ($\mathbb{R}^2$); however, another variant investigates the chromatic number of the rational plane, denoted by $\chi(\mathbb{Q}^2)$. This problem is solved, and the result is as follows:

Theorem 8.1. $\chi(\mathbb{Q}^2) = 2$

This is a more complicated result to prove, and two different methods are described below. The first [1] separates the plane into classes that can be translated onto each other and proves a 2-coloring for one of the classes. The second method [7] proves that a polygon in the rational plane with side lengths of 1 must have an even number of sides.

Before we begin the first proof, let's consider the set of integers ($\mathbb{Z}^2$). Proving that the chromatic number of this graph is 2 is very similar to our first proof for the rational plane, but it's much simpler and more intuitive.

Proof. Consider the plane of all integers ($\mathbb{Z}^2$). A unit-distance graph in this plane would contain exclusively edges that are parallel to either the x or y axis. We could

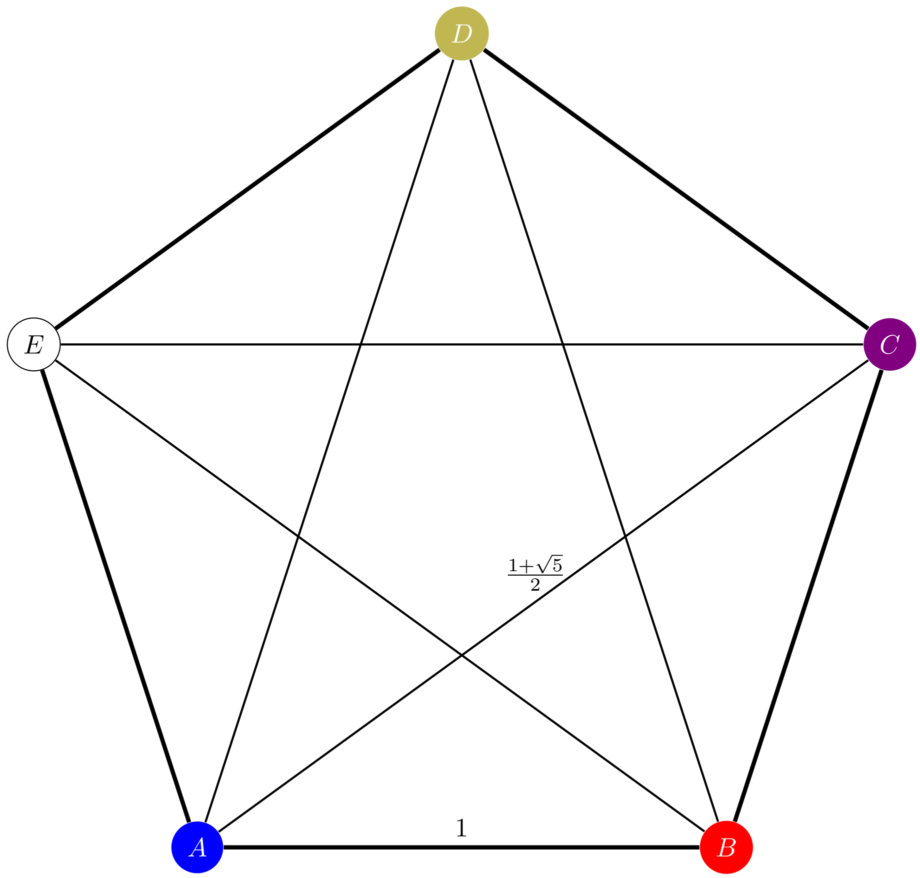

FIGURE 8. Pentagon with four colors

not include any diagonal lines in this graph because the shortest diagonal between two integers would have a length of $\sqrt{2}$, which is longer than the required length of 1. Figure 9 shows a finite subset of the integer plane with all its vertices and edges.

Now, coloring this graph with two colors is very simple. The vertices must be a different color than the 4 vertices surrounding them distance 1 away, but they may be the same color as the 4 vertices on a diagonal from them. Therefore, we color the graph in a sort of checkerboard pattern.

Let o equal some odd integer and e equal some even integer (not necessarily the same number each time it is mentioned). If the coordinates of a point follow the format (o, o) or (e, e), then we color it red. If the coordinates follow the format (e, o) or (o, e), then we color it blue. As we can see in Figure 10, no two points of the same color are at distance 1 apart. We can use this same method of coloring the plane depending on if they're odd or even to prove that the rational plane has a chromatic number of 2. Now, with this in mind, we will begin this proof.

□

Proof. For this proof, we partition the rational plane into classes (similar to how we used the de Bruijn-Erdős Theorem in the original problem to focus on finite sets

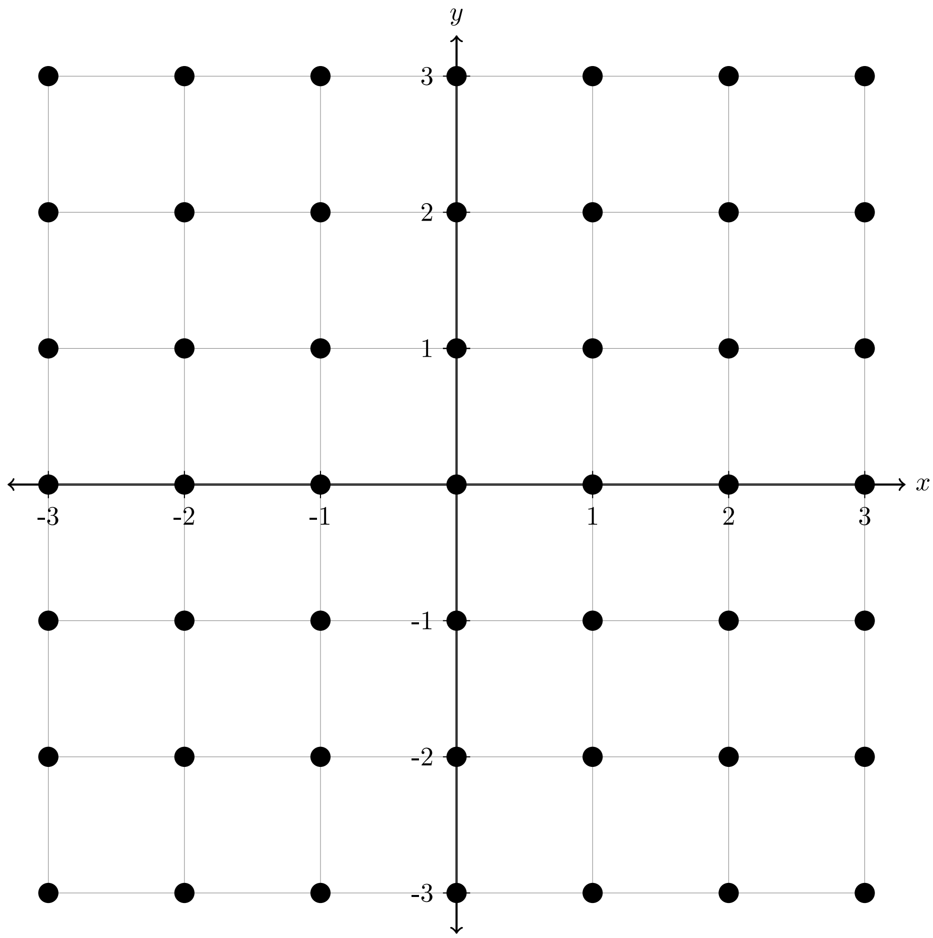

FIGURE 9. Unit-distance graph on the integer plane

of the plane). We will prove an equivalence relation between these classes and then prove a coloring that works for all of them.

Definition 8.2 (Equivalence relation)**.** An equivalence relation, denoted by " ", exists between elements of a set (called an equivalence class), all with some aspect in common, such that all elements are labeled as being the "same". Brackets are used to refer to a class containing an element (for example, to refer to the class containing the origin, we can write "[(0,0)]"). Equivalence relations have the three following properties:

(1) $x \sim x$.
(2) If $x \sim y$, then $y \sim x$.
(3) If $x \sim y$ and $y \sim z$, then $x \sim z$.

For this proof, we will define an equivalence relation between two points if both the distance between their x coordinates and the distance between their y coordinates have an odd denominator when written as a fraction $\frac{p}{q}$ in lowest terms. If

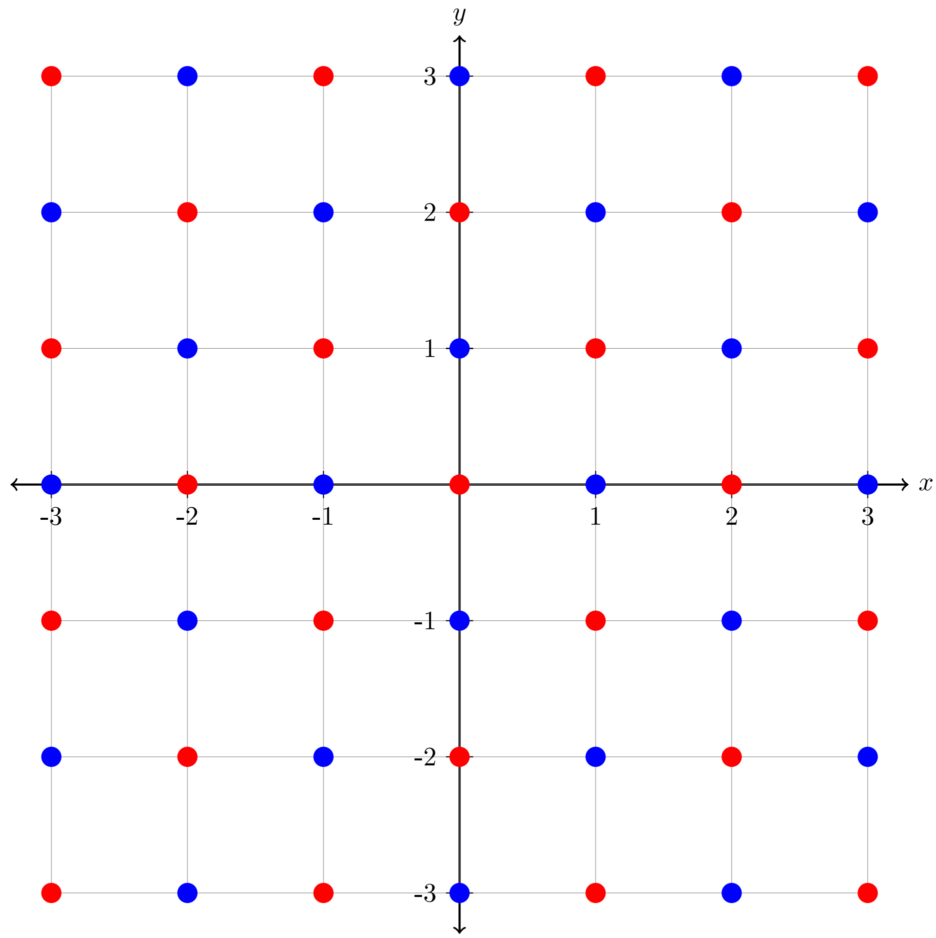

FIGURE 10. The chromatic number of the integer plane is 2

two points are at unit distance apart, they will be in the same class (if one or both of the differences between the x and y coordinates had an even denominator when written in simplest terms, the distance between the points couldn't be 1).

Consider the rational points (r_1, r_2) and (q_1, q_2) at distance 1. We can set $(r_1 - r_2)$ to $\frac{a}{b}$ and $(q_1 - q_2)$ to $\frac{c}{d}$, with b and d being odd (since we stated above that $(r_1 - r_2)$ and $(q_1 - q_2)$ would have odd denominators). Now, we have the following:

$$(r_1 - q_1)^2 + (r_2 - q_2)^2 = 1$$
$$(\frac{a}{b})^2 + (\frac{c}{d})^2 = 1$$
$$a^2d^2 + b^2c^2 = b^2d^2$$

Consider [(0,0)], the class containing the origin. We will use the same method to color these points that we did for the integers earlier. In fact, since this class contains the origin, the only difference here is that the edges can make triangles using Pythagorean triples, whereas this didn't work with the integers. We color the rational points red if their coordinates follow the format $(\frac{o}{o}, \frac{o}{o})$ or $(\frac{e}{o}, \frac{e}{o})$. We color the points blue if their coordinates follow the format $(\frac{e}{o}, \frac{o}{o})$ or $(\frac{o}{o}, \frac{e}{o})$. If we plug in these values for (r_1, r_2) and (q_1, q_2), we can see that no two points of the same color can lie at unit distance apart.

Now, we want to prove that all classes are the same shape and can therefore be colored in the same way so that proving the chromatic number of of one class applies to the entire rational plane.

Since we have proved that $\chi(\mathbb{Q}^2) = 2$ for [(0,0)], we now want to prove that translations exist to bijectively map the class containing the origin onto any other class. We will say that [(0,0)] contains the point $(\frac{x}{y}, \frac{w}{z})$; therefore, y and z must be odd. We will also consider $[(\frac{a}{b}, \frac{c}{d})]$ where b and d are odd. We want to prove that there is a one-to-one translation to map [(0,0)] to $[(\frac{a}{b}, \frac{c}{d})]$. To map [(0,0)] onto $[(\frac{a}{b}, \frac{c}{d})]$, we consider the point $(\frac{x}{y} + \frac{a}{b}, \frac{w}{z} + \frac{c}{d})$. To prove that this point is in the same class as $(\frac{a}{b}, \frac{c}{d})$, we can subtract the x and y values of these two points:

$$(\frac{x}{y} + \frac{a}{b}) - (\frac{a}{b}) = \frac{x}{y}$$
$$(\frac{w}{z} + \frac{c}{d}) - (\frac{c}{d}) = \frac{w}{z}$$

We already know that y and z are odd; therefore, the differences between the x and y coordinates of these two points are odd and must be in the same class. Because of this, we conclude that [(0,0)] can be mapped bijectively onto any other class, and then we can color every other class using the same method we did to prove that the chromatic number of $[(0, 0)] = 2$. Therefore, the chromatic number of the rational plane is 2.

□

Now we will look at a second proof that approaches this problem in a different way. This proof is longer, but it may be easier to follow.

Proof. We can also look at this problem by considering a polygon in the rational plane. A graph is bipartite if and only if it contains no odd cycle.

Definition 8.3 (Bipartite graph)**.** A graph G that can be divided into two nonempty sets A and B such that each edge in G has one endpoint in A and one in B. [8].

Therefore, a bipartite graph would have a chromatic number of 2 since we can color the points in set A red and the points in set B blue. Since each edge has one endpoint in each set, no two adjacent points would have the same color.

If a polygon has an even number of sides, then it has an even number of vertices, and we can alternate between two colors for each vertex. However, a polygon with an odd number of sides has an odd number of vertices and cannot be colored this way with only two colors (see Figure 11). We can prove the chromatic number by demonstrating that any polygon in the rational plane ($\mathbb{Q}^2$) with equal side lengths must have an even number of sides.

Consider a polygon in the rational plane with n vertices and n congruent sides. We can label the coordinates of this polygon $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Figure 12 shows a polygon with equal side lengths where $n = 6$.

Now, since the distance between two adjacent vertices is 1, we can use the distance formula to say that $\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = 1$. Since this is in the

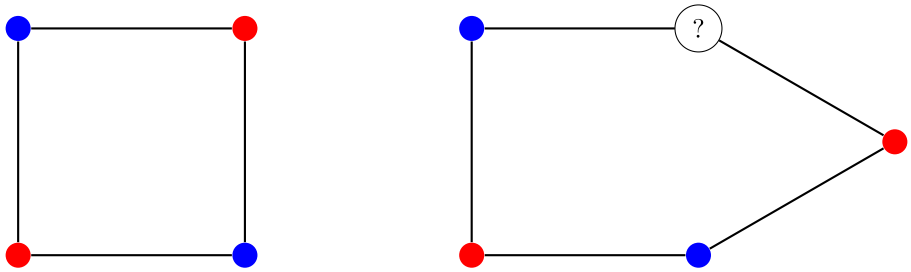

FIGURE 11. A polygon with an even number of sides; a polygon with an odd number of sides

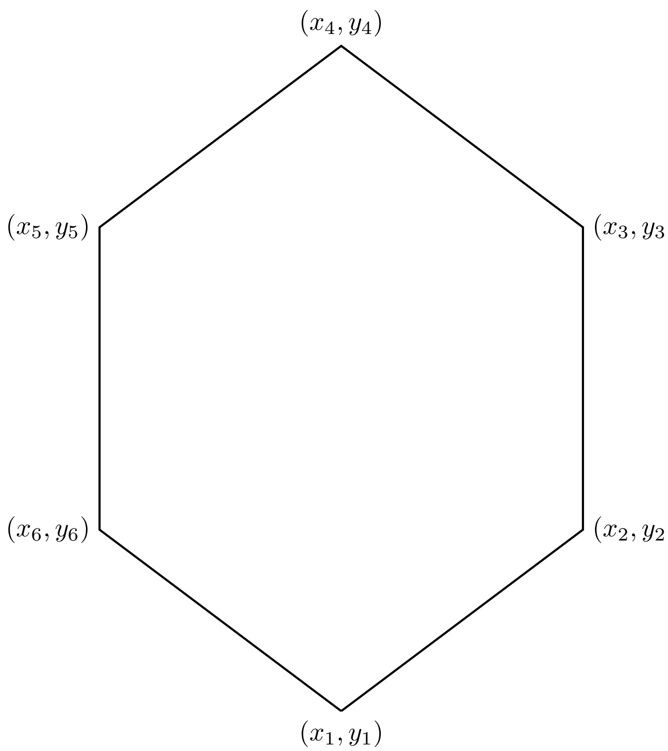

FIGURE 12. A polygon with six congruent sides

rational graph, both $x_{i+1} - x_i$ and $y_{i+1} - y_i$ can be represented by a fraction $\frac{p}{q}$. We will set $x_{i+1} - x_i$ to $\frac{a_i}{c_i}$ and $y_{i+1} - y_i$ to $\frac{b_i}{c_i}$ (using the common denominator c when writing these fractions). Figure 13 shows how $\frac{a_4}{c_4}$ and $\frac{b_4}{c_4}$ represent the distances between adjacent x and y coordinates, respectively. Now, we can plug these values into the distance formula and simplify it:

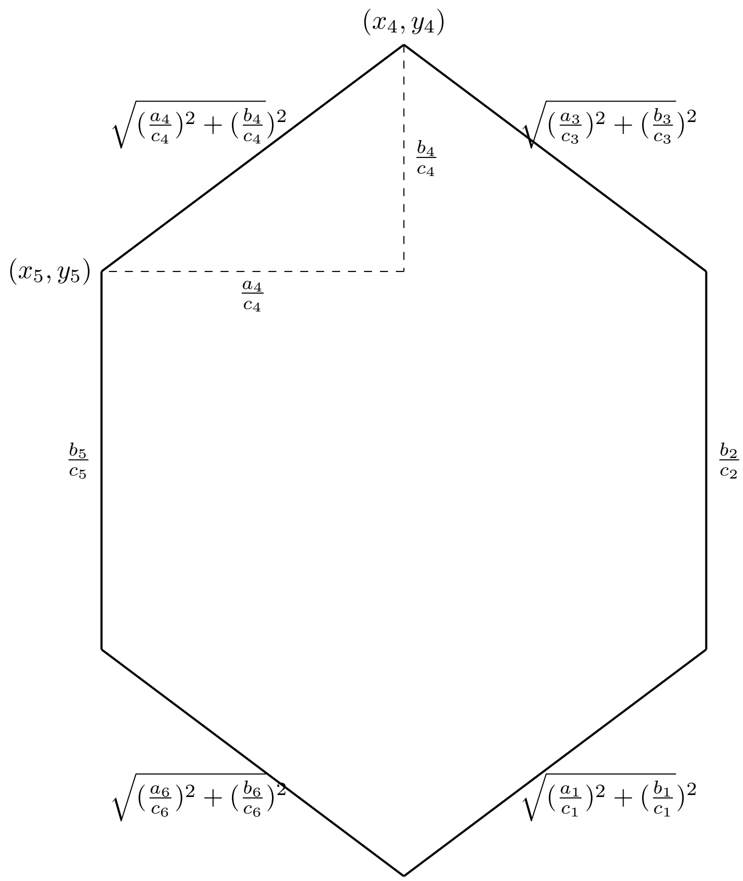

FIGURE 13. The roles of a, b, and c in the polygon

$$\begin{aligned} \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} &= 1 \\ \sqrt{(\frac{a}{c})^2 + (\frac{b}{c})^2} &= 1 \\ \frac{a^2}{c^2} + \frac{b^2}{c^2} &= 1 \\ a^2 + b^2 &= c^2 \end{aligned}$$

Since all the vertices and distances must be rational and the values of a, b, and c must be integers, one of two cases are possible for their values. The first is that they are a Pythagorean triple. In Figure 13), (x_4, y_4) and (x_5, y_5) are two vertices of the right triangle shown in dashed lines, so a, b, and c must be a Pythagorean triple. The second case is that or a or b is 0. Between points (x_5, y_5) and (x_6, y_6), there is no change in the x value; therefore, if we tried to create a right triangle to determine the distance between the points, a_5 would be equal to 0, and b_5 and c_5

would both be equal to 1. This is true for edges that are parallel to either axis of the coordinate plane.

When the first case is true and a, b, and c are a Pythagorean triple, c must be odd, and either a or b is also odd and the other is even. This is because Pythagorean triples always consist of an odd square being the sum of an even square and an odd square; we cannot add two odd squares and obtain an even square. Also, all three values cannot be even (a multiple of a Pythagorean triple), since the fractions $\frac{a}{c}$ and $\frac{b}{c}$ would not be in their lowest terms. When the second case is true and a or b is 0, we have the same situation here. One of the values must be odd (whichever is equal to 1) and the other must be even (whichever is equal to 0). The important takeaway from this part of the proof is that a and b are never even or odd at the same time; for any distance between two vertices, one of them must be even and the other must be odd. Therefore, the number of odd a values is equal to the number of even b values, and the number of even a values is equal to the number of odd b values.

In the next part of this proof, we will be adding up all the values of $\frac{a}{c}$ and then doing the same for $\frac{b}{c}$. Since the values for c are different for each distance between two adjacent vertices, we must multiply all the fractions $\frac{a}{c}$ and $\frac{b}{c}$ by a scalar to find a common denominator for all these values. However, since all values of c are odd, this scalar will be an odd number o over itself $\frac{o}{o}$. Therefore, the new common denominator c' will still be odd. In the numerators, odd values of a and b will also remain odd (these new values are denoted a' and b'), and even values will remain even.

Now, if we were to add up all the values of a' or all the values of b', we would end up with 0. Since this is a closed figure, we can start from any vertex and go around the figure, adding up the differences in x and y coordinates. Since we would end up at the same vertex, the sums of these values must come out to 0. Because of this, there cannot be an odd number of odd a' or b' values. For example, if the values of a for the entire polygon included an odd number of odd values for a', the total sum couldn't be 0. Therefore, there must be an even number of odd values of both a' and b'.

By now, we have proved the following statements:

(1) The number of odd a values is equal to the number of even b values.
(2) The number of even a values is equal to the number of odd b values.
(3) There are an even number of odd a values.
(4) There are an even number of odd b values.

If we consider statements (1) and (3), we can conclude that there must be an even number of even b values. If we consider statements (2) and (4), we can conclude that there must be an even number of even a values. Therefore, there is an even number of all values of both a and b. Since each value of a and b represent one vertex of the polygon, the graph must contain an even number of points and therefore has a chromatic number of 2.

□

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