Elliptic Integrals - IRPW

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1 Abstract

Elliptic integrals are mathematical functions that include the integration of combinations of polynomials and square roots of cubic or quartic functions, and these integrals cannot be stated using fundamental functions. These integrals have been an integral part of mathematical analysis since their introduction in the 18th century. Elliptic functions play a crucial role in several applications, particularly in physics and engineering. These functions are essential for addressing complex issues such as pendulum motion and electrical circuit designs. The classification of mathematicians such as Fagnano, Euler, Gauss, and Legendre into canonical forms is the product of their significant efforts. Modern computer tools, including transformations and series expansions, are also used in this process. This paper aims to explore the historical background of elliptic integrals, their complex features, and computational techniques.

2 Introduction

A general form of an elliptic integral can be represented as:

$$E(x) = \int R\left(x, \sqrt{P(x)}\right)\, dx,$$

where R is a rational function of its arguments, and $P(x)$ is a polynomial of degree three or four.

Elliptic integrals possess a profound level of complexity, making them inherently challenging to compute and apply due to their intricate nature.

There exist three major types of elliptic integrals. They are distinguished according to the form of their integrands and to the field of their applications:

2.1 Elliptic Integrals of the First Kind

The elliptic integral of the first kind, denoted by $F(\phi, k)$, is represented as:

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

where ϕ is the amplitude and k $(0 \leq k < 1)$ is the modulus.

In terms of geometry, this integral represents the arc length of an ellipse. The function $F(\phi, k)$ an be alternatively seen as a mapping from amplitude ϕ and modulus k to a real number, effectively converts the angular parameterization by introducing a length parameter.

In the special case where $\phi = \frac{\pi}{2}$, the integrand reduces to the complete elliptic integral of the first kind, denoted by $K(k)$:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

2.2 Elliptic Integrals of the Second Kind

The elliptic integral of the second kind, denoted by $E(\phi, k)$, is defined as:

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

This integral also covers the arc length of the ellipse, though differently from the integral of the first kind. In this case, the integrand contains a square root of $1 - k^2 \sin^2 \theta$, which changes the weight of the integrand over the integration interval.

For the complete elliptic integral of the second kind, we have:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

2.3 Elliptic Integrals of the Third Kind

The elliptic integral of the third kind, denoted by $\Pi(n; \phi, k)$, involves an additional parameter n known as the characteristic, and is defined as:

$$\Pi(n; \phi, k) = \int_0^{\phi} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.$$

This integral generalizes the first and second kinds by incorporating a pole at $\theta = \arcsin(1/n)$ if $n > 1$. The parameter n allows for the elliptic integral of the third kind to address more complicated geometries and physical situations.

For the complete elliptic integral of the third kind:

$$\Pi(n; k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.$$

For a class of addition equations to be true, elliptic integrals have some pleasant properties: they are periodic and symmetric, therefore allowing a complex integral to be reduced into multiple simple ones. These addition equations also include correction terms; the mentioned correction terms in them reflect the outcomes of the interaction of the integrals upon their combination.

For the first kind, the addition formula can be expressed as:

$$F(\phi_1 + \phi_2, k) = F(\phi_1, k) + F(\phi_2, k) + \text{correction terms}$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind:

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

The correction terms in this addition formula can be derived based on the properties of Jacobi's elliptic functions. Specifically, the correction term involves the sine and cosine of the angles being summed and the elliptic modulus k. The explicit correction term is:

$$\frac{k^2 \sin\phi_1 \sin\phi_2 \sin(\phi_1+\phi_2)}{1-k^2\sin^2\phi_1\sin^2\phi_2}$$

In certain limits, elliptic integrals can be turned into simpler forms. When $k = 0$:

$$F(\phi, 0) = \phi$$
$$E(\phi, 0) = \phi$$

Here, $E(\phi, k)$ is the incomplete elliptic integral of the second kind:

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta}\, d\theta$$

As k approaches 1, the integrals exhibit logarithmic behavior. For instance, the complete elliptic integrals of the first kind $K(k)$ and the second kind $E(k)$ behave as:

$$K(k) \sim -\frac{1}{2}\ln(4(1-k))$$
$$E(k) \sim 1 + \frac{1}{2}\ln(4(1-k))$$

To derive the addition formula and the correction terms for the first kind, we can use the properties of the Jacobi elliptic functions. The incomplete elliptic integral of the first kind can be expressed in terms of the Jacobi elliptic function $\mathrm{sn}(u, k)$, where $u = F(\phi, k)$ and $\phi = \mathrm{am}(u, k)$. Here, $\mathrm{am}(u, k)$ is the Jacobi amplitude.

Using the addition formulas for Jacobi elliptic functions, we can write:

$$\mathrm{sn}(u_1 + u_2, k) = \frac{\mathrm{sn}(u_1, k)\mathrm{cn}(u_2, k)\mathrm{dn}(u_2, k) + \mathrm{sn}(u_2, k)\mathrm{cn}(u_1, k)\mathrm{dn}(u_1, k)}{1 - k^2\mathrm{sn}^2(u_1, k)\mathrm{sn}^2(u_2, k)}$$

When we put these functions back into their form with ϕ, we get the correction term for the addition formula.

3 History of Elliptic Integrals

The theory of elliptic integrals began largely in the 18th century and began to develop rapidly due to the work of many leading mathematicians. This section will provide a rather detailed history of the development of the theory of elliptic integrals, at that time with an emphasis on mathematical accuracy, and a plethora of equations that came into circulation.

The foundations of elliptic integral theory were laid by the work of Giulio Carlo Fagnano, 1682–1766. He studied the lemniscate, whose Cartesian equation is:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

and in polar coordinates as:

$$r^2 = a^2 \cos 2\theta,$$

led to the exploration of the lemniscatic integral:

$$s(r) = \int_0^r \frac{dt}{\sqrt{1 - t^4}}.$$

The lemniscatic integral gives the arc length of the lemniscate from the origin to a point on the curve. This integral is expressed as $s(r) = \int_0^r \frac{dt}{\sqrt{1-t^4}}$. The integral stands for the sum of infinitesimal portions of the arc length, each segment being $\frac{dt}{\sqrt{1-t^4}}$ long. When these segments are added from 0 to r, the total arc length of the lemniscate from the origin to the point at r is obtained.

A major contribution of Fagnano's was in how he was able to deal with these integrals. For instance, he demonstrated that:

$$\int \frac{dr}{\sqrt{1 - r^4}} = \sqrt{2} \int \frac{dt}{\sqrt{1 + t^4}},$$

using the substitution:

$$t = \frac{1}{r}\sqrt{1 \pm \sqrt{1 - r^4}}.$$

And these reductions were the bases for further developments.

Leonhard Euler, 1707–1783, followed with his own work on Fagnano, and he pushed the elliptical integrals to a more established field. He made a lot of change in the field of ellipticials, to which Euler made contributions including the formulation of addition theorems:

$$\int_0^u \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = \int_0^w \frac{dt}{\sqrt{1-t^4}},$$

where u and v are related by:

$$w = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}.$$

The importance of such addition formulae to the development of the theory of elliptic integrals is truly comparable to that of addition formulae for trigonometric functions. These developments established the modern theory of elliptic functions and their numerous applications in mathematics and physics.

Adrien-Marie Legendre worked much on elliptic integrals. He was the first to systematize them and proposed dividing them into three types, which are now named Legendre's forms.

Among the works of Legendre are those developing properties and series expansions for these integrals. The complete elliptic integral of the first kind is one of the significant examples, which is normally denoted by $K(k)$. This integral is defined as:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Legendre also developed a series expansion for $K(k)$. This suggests a representation of the integral as an infinite series. The series expansion for $K(k)$ is:

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n}.$$

Here, $(2n-1)!!$ and $(2n)!!$ denote double factorials. The double factorial of a number n is the product of all the integers from 1 to n that have the same parity (odd or even) as n. For example:

The double factorial of an odd number: $(2n-1)!! = (2n-1)\cdot(2n-3)\cdot\dots\cdot 3\cdot 1$.

The double factorial of an even number: $(2n)!! = (2n)\cdot(2n-2)\cdot\dots\cdot 4\cdot 2$.

To learn more about where this series expansion comes from, let's consider the definition of the integral of $K(k)$. This integral is of the form that is difficult to integrate directly, and he instead used a method called binomial series expansion.

First, we write the integrand in a form that makes it easier to expand:

$$\frac{1}{\sqrt{1-k^2\sin^2\theta}} = (1-k^2\sin^2\theta)^{-\frac{1}{2}}.$$

Using the binomial series expansion for $(1-x)^{-\frac{1}{2}}$, we get:

$$(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x)^n.$$

Here, $\binom{-\frac{1}{2}}{n}$ is a generalized binomial coefficient. For our case:

$$\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!}.$$

Simplifying this, we get:

$$\binom{-\frac{1}{2}}{n} = (-1)^n\frac{(2n-1)!!}{2^n n!}.$$

Substituting back into the series expansion, we get:

$$(1 - k^2 \sin^2 \theta)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} (k^2 \sin^2 \theta)^n.$$

Now, integrating term-by-term from 0 to $\frac{\pi}{2}$:

$$K(k) = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} k^{2n} \sin^{2n} \theta \, d\theta.$$

Interchanging the sum and integral, which is allowed in a particular sense, we obtain:

$$K(k) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, d\theta.$$

The integral $\int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, d\theta$ can be evaluated using the following known formula for definite integrals of sine functions::

$$\int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, d\theta = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}.$$

Thus, substituting this back into our series, we get:

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 k^{2n}.$$

It follows to complete the proof of series expansion of the complete elliptic integral of the first kind $K(k)$. This expansion is useful because it allows us to approximate $K(k)$ for small values of k and gives an insight into the behavior of elliptic integrals.
Carl Gustav Jacob Jacobi,1804–1851, extended the theory of elliptic integrals by introducing the Jacobi elliptic functions: $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, and $\mathrm{dn}(u, k)$. Jacobi worked on the inversion of elliptic integrals and succeeded in discovering a major property about them.
Jacobi's inversion problem involves solving for u in terms of ϕ and k. This is essential to the understanding of elliptic integrals, as it turns what was an ungainly elliptic integral into more tractable functions.

The Jacobi elliptic functions are defined as follows:

$$
\begin{aligned}
\mathrm{sn}(u,k) &: \text{the sine amplitude function,} \\
\mathrm{cn}(u,k) &: \text{the cosine amplitude function,} \\
\mathrm{dn}(u,k) &: \text{the delta amplitude function.}
\end{aligned}
$$

These functions relate to the elliptic integral of the first kind $F(\phi, k)$:

$$F(\phi,k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

When you solve this integral for ϕ, you get:

$$u = F(\phi, k).$$

The inverse problem is to find ϕ given u:

$$\phi = \mathrm{am}(u,k),$$

where $\mathrm{am}(u,k)$ is the amplitude function. The Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ are then defined by:

$$
\begin{aligned}
\mathrm{sn}(u,k) &= \sin(\mathrm{am}(u,k)), \\
\mathrm{cn}(u,k) &= \cos(\mathrm{am}(u,k)), \\
\mathrm{dn}(u,k) &= \sqrt{1-k^2\sin^2(\mathrm{am}(u,k))}.
\end{aligned}
$$

With those functions we can do more work with elliptic integrals, since it uses a back-way to invert them.

Karl Weierstrass (1815–1897) further developed the theory by introducing the Weierstrass elliptic function $\wp(z)$, along with related functions $\zeta(z)$ and $\sigma(z)$. The work of Weierstrass unified elliptic integrals and functions with the help of complex analysis, and the theoretical background of this field was enriched.

The Weierstrass elliptic function $\wp(z)$ is defined by:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where Λ is a lattice in the complex plane, and Λ^* is the lattice excluding the origin.

Similarly, such a function is doubly periodic, expressing elliptic integral behavior over a lattice of complex numbers.

3.1 Equations and Series Expansions

Elliptic integrals can be expressed using series expansions, which are important for their numerical evaluation.

To explore the series expansions of the complete elliptic integrals of the second and third kinds, and understand how these series are derived, first, consider the complete elliptic integral of the second kind, $E(k)$:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

To derive the series expansion for $E(k)$, we can use a technique similar to the binomial series expansion. Let's rewrite the integrand in a more convenient form:

$$\sqrt{1 - k^2 \sin^2 \theta} = (1 - k^2 \sin^2 \theta)^{\frac{1}{2}}.$$

Now, we can use the binomial series expansion for $(1-x)^{\frac{1}{2}}$:

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 - \cdots .$$

By substituting $x = k^2 \sin^2 \theta$, we get:

$$\sqrt{1 - k^2 \sin^2 \theta} = 1 - \frac{1}{2} k^2 \sin^2 \theta - \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \theta - \cdots .$$

Now, we integrate term by term from 0 to $\frac{\pi}{2}$:

$$E(k) = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{2} k^2 \sin^2 \theta - \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \theta - \cdots \right) d\theta.$$

Each term in the series can be integrated separately:

$$E(k) = \int_0^{\frac{\pi}{2}} d\theta - \frac{k^2}{2} \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta - \frac{3k^4}{32} \int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta - \cdots .$$

We use the following integrals for the sine functions:

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta = \frac{\pi}{4}, \quad \int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta = \frac{3\pi}{16}.$$

Substituting these values, we get:

$$E(k) = \frac{\pi}{2} - \frac{k^2}{2} \cdot \frac{\pi}{4} - \frac{3k^4}{32} \cdot \frac{3\pi}{16} - \cdots .$$

Simplifying, we get the series expansion for $E(k)$:

$$E(k) = \frac{\pi}{2} \left(1 - \frac{k^2}{4} - \frac{3k^4}{64} - \frac{5k^6}{256} - \cdots \right).$$

Next, consider the complete elliptic integral of the third kind, $\Pi(n, k)$:

$$\Pi(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}.$$

To find the series expansion for $\Pi(n,k)$, we can use a perturbation approach, assuming n is small. For small n, the integrand can be expanded as:

$$\frac{1}{1-n\sin^2\theta} \approx 1 + n\sin^2\theta + n^2\sin^4\theta + \cdots.$$

Substituting this into the integral, we get:

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \left(1 + n\sin^2\theta + n^2\sin^4\theta + \cdots\right) \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

This can be separated into individual integrals:

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + n\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}} + \cdots.$$

The first term is just the complete elliptic integral of the first kind $K(k)$:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = K(k).$$

The second term involves the complete elliptic integral of the second kind $E(k)$. Using the relation for integrals involving $\sin^2\theta$:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}} = K(k) - E(k).$$

Thus, the series expansion for $\Pi(n,k)$ is:

$$\Pi(n,k) = K(k) + \frac{n}{4}\left(K(k) - E(k)\right) + \cdots.$$

These series converge rapidly for small values of k and n, making them useful for practical computations. By breaking down the integrals into simpler parts, we can approximate the values of these elliptic integrals more easily.

4 Computational Methods

One of the important ways to find the values of elliptic integrals is to find answers to solve the problems of modern analysis, especially when analytical answers are very hard to find and, moreover, impossible. This chapter goes in-depth about numerical methods such as Gaussian quadrature, numerical integration techniques like Romberg integration, and the Arithmetic-Geometric Mean method. There are mathematics formulas and ways of their applications.

4.1 Gaussian Quadrature

Gaussian quadrature is an integration technique for numerical integrals to approximate a function's integral as the sum of values of this function at certain points within the domain of integration. The collection of points and weights are chosen such that, for example, the rule integrates exactly all polynomials of the highest possible grade, or degree of exactness.

For an elliptic integral of the form

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

Gaussian quadrature can be applied by transforming the integral into a suitable form. Typically, this involves mapping the integral to the interval $[-1, 1]$ using a change of variables, and then applying the quadrature rule. The standard Gaussian quadrature rule is given by:

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where x_i are the quadrature points (roots of the Legendre polynomial $P_n(x)$) and w_i are the corresponding weights. For example, for $n = 2$,

$$\int_{-1}^{1} f(x)\, dx \approx w_1 f(x_1) + w_2 f(x_2),$$

with $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$, $w_1 = w_2 = 1$.

To apply Gaussian quadrature to the elliptic integral, first, we need to express it in the form:

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

Letting $\theta = \frac{\phi}{2}(1 + x)$, the integral becomes:

$$F(\phi, k) = \frac{\phi}{2} \int_{-1}^{1} \frac{1}{\sqrt{1 - k^2 \sin^2 \left(\frac{\phi}{2}(1 + x) \right)}} \, dx.$$

Applying Gaussian quadrature:

$$F(\phi, k) \approx \frac{\phi}{2} \sum_{i=1}^{n} w_i \frac{1}{\sqrt{1 - k^2 \sin^2 \left(\frac{\phi}{2}(1 + x_i) \right)}}.$$

4.2 Romberg Integration

Romberg integration is an algorithm that gives successive improvement in the accuracy of the trapezoidal rule by means of Richardson's extrapolation. The definition of an elliptic integral, such as

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta,$$

the trapezoidal rule may then be used for an approximate evaluation of the integral over an initial partition of the interval $[0, \phi]$:

$$T_1 = \frac{\phi}{2} [f(0) + f(\phi)].$$

Refinement of partition by halving the step size results in more accurate estimate:

$$T_2 = \frac{\phi}{4} \left[f(0) + 2f \left(\frac{\phi}{2} \right) + f(\phi) \right].$$

Romberg integration refines this estimate by using the result of the trapezoidal rule with different step sizes and then applying Richardson extrapolation:

$$R_{k,j} = R_{k-1,j} + \frac{R_{k-1,j} - R_{k-1,j-1}}{4^j - 1},$$

where $R_{k,j}$ is the Romberg estimate after k refinements and j levels of extrapolation. This method goes on progressively to reduce error through refinement and extrapolation.

4.3 Arithmetic-Geometric Mean (AGM)

The Arithmetic-Geometric Mean (AGM) method is another powerful technique for computing elliptic integrals. The convergence of the AGM can be demonstrated by noting that both sequences $\{a_n\}$ and $\{b_n\}$ are bounded and monotonic. Since $a_{n+1} \leq a_n$ and $b_{n+1} \geq b_n$, and both sequences are bounded by a_0 and b_0, they converge to a common limit $M(a_0, b_0)$. Formally, we can write:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a_0, b_0).$$

The AGM iteration can be effectively applied to compute the complete elliptic integrals. We start with the initial values:

$$a_0 = 1, \quad b_0 = \sqrt{1-k^2}, \quad c_0 = k.$$

The iteration process is then given by:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad c_{n+1} = \frac{a_n - b_n}{2}.$$

This process is repeated until a_n and b_n converge. Let a_∞ be the common limit of the sequences $\{a_n\}$ and $\{b_n\}$.

The complete elliptic integral of the first kind $K(k)$ can be computed using the AGM as follows:

$$K(k) = \frac{\pi}{2a_\infty}.$$

Firstly, start with the AGM iterations for a_n and b_n:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Then, both sequences converge to the AGM $M(a_0, b_0)$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a_\infty.$$

The integral form of the complete elliptic integral is:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

Using the AGM method, we can express $K(k)$ as:

$$K(k) = \frac{\pi}{2M(1, \sqrt{1-k^2})}.$$

Since $a_0 = 1$ and $b_0 = \sqrt{1-k^2}$, it follows that:

$$K(k) = \frac{\pi}{2a_\infty}.$$

The complete elliptic integral of the second kind $E(k)$ can be computed using the AGM and the sequence $\{c_n\}$:

$$E(k) = \frac{\pi}{2a_\infty} \left(1 - 2\sum_{n=0}^{\infty} 2^{-n} c_n^2 \right).$$

To compute this sequence, first, start with the initial values and iterations:

$$a_0 = 1, \quad b_0 = \sqrt{1-k^2}, \quad c_0 = k.$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad c_{n+1} = \frac{a_n - b_n}{2}.$$

The sequences $\{a_n\}$ and $\{b_n\}$ converge to a_∞:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a_\infty.$$

The sequence $\{c_n\}$ is used to adjust the calculation for the second kind integral. The formula for $E(k)$ is derived by considering the additional terms from c_n:

$$E(k) = \frac{\pi}{2a_\infty}\left(1 - 2\sum_{n=0}^{\infty} 2^{-n} c_n^2\right).$$

5 Advanced Topics

Elliptic integrals generalize widely to more sophisticated mathematical topics, most notably, to complex analysis, and to relationships with certain special functions. This module takes a look at these more subtle features of the theory, with special emphasis on methods of contour integration, many applications to complex differential equations, and the relationship to hypergeometric functions and elliptic functions.

In the subject of complex analysis, elliptic integrals can be adopted with the contour integration techniques and have the considerable application to the solution of complex differential functions.

5.0.1 Contour Integration Methods

Contour integration is a significant method in complex analysis for the computation of integrals along arbitrary paths in the complex plane. The applicability of contour integration in handling integrals involving branch cuts and poles is in integrals for elliptic functions.

Consider the elliptic integral of the first kind:

$$F(z,k)=\int_0^z \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

By transforming this integral into the complex plane, we can utilize residues and contour deformations to evaluate it more efficiently. For example, if we take the substitution $\theta=\mathrm{am}(u,k)$, where $\mathrm{am}(u,k)$ is the Jacobi amplitude function, we can rewrite the integral as:

$$F(z,k)=\int_0^{\mathrm{am}(z,k)} \frac{d(\mathrm{am}(u,k))}{\sqrt{1-k^2\sin^2(\mathrm{am}(u,k))}}.$$

In the complex plane, we consider a contour that encircles the branch points of the integrand. By applying the residue theorem, we can evaluate the integral by summing the residues at the poles within the contour:

$$\oint_\Gamma \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=2\pi i\sum \mathrm{Res}\left(\frac{1}{\sqrt{1-k^2\sin^2\theta}},\theta_j\right),$$

where θ_j are the poles of the integrand inside the contour Γ.

5.0.2 Applications to Complex Differential Equations

Elliptic integrals naturally come out as solutions of complex differential equations. Among them are the differential equations which define elliptic functions, such as Jacobi elliptic functions. Consider the following differential equation:

$$\frac{d^2y}{dz^2}=(a-2b\cos(2y))\,y,$$

where a and b are constants. The solutions to such equations can often be expressed using elliptic integrals. For instance, by making the substitution $y=\mathrm{sn}(u,k)$, where $\mathrm{sn}(u,k)$ is the Jacobi elliptic sine function, we can transform the differential equation into:

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2),$$

which is directly related to the elliptic integral of the first kind:

$$u = F(\sin^{-1}(y), k).$$

5.1 Connections with Special Functions

Elliptic integrals are closely related to several classes of special functions, including hypergeometric functions and elliptic functions.

5.1.1 Hypergeometric Functions

Elliptic integrals can be expressed in terms of hypergeometric functions. For example, the incomplete elliptic integral of the first kind can be written as:

$$F(\phi, k) = \phi \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2 \sin^2\phi\right),$$

where ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function defined by the series:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(a)_n$ being the Pochhammer symbol (rising factorial):

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

This bond supplies a channel to move from elliptic integrals into relationships containing hypergeometric functions, or making use of some hypergeometric series and transformations on the related function in order to evaluate or provide an approximation for an elliptic integral.

5.1.2 Elliptic Functions and Modular Forms

Elliptic integrals are intimately connected with elliptic functions, such as the Weierstrass $\wp$-function and Jacobi elliptic functions. These functions can be defined via the inversion of elliptic integrals. For instance, the Jacobi elliptic function $\mathrm{sn}(u, k)$ is defined as the inverse of the incomplete elliptic integral of the first kind:

$$u = F(\mathrm{sn}(u, k), k).$$

The differential equation satisfied by $\mathrm{sn}(u, k)$ is:

$$\left(\frac{d}{du}\mathrm{sn}(u, k)\right)^2 = (1 - \mathrm{sn}^2(u, k))(1 - k^2\mathrm{sn}^2(u, k)),$$

which directly ties back to the elliptic integral.

The modular forms, which are functions of the complex upper half-plane transforming under the action of the modular group in certain ways, reside together with elliptic integrals. The periods of elliptic integrals define lattices in the complex plane used in the construction of modular forms. For instance, the Eisenstein series $E_k(\tau)$, a type of modular form, can be expressed in terms of lattice sums involving elliptic integrals:

$$E_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m + n\tau)^k},$$

where τ is in the upper half-plane.

The following advanced topics demonstrate deep connections and applications to complex analysis and special functions, therefore making elliptic integrals fundamentally important in mathematical theory.

6 Conclusion

Ever since their discovery in the 18th century, elliptic integrals have formed the center of mathematical analysis and have had numerous applications. The great contribution of the mathematicians Fagnano, Euler, Legendre, Jacobi, and Weierstrass have been on the foundation of contemporary understanding and classification of their canonical forms.

Defined through integrals with respect to polynomials and cubic or quartic functions square roots, elliptic integrals are surpassed in importance to any of the elementary functions and are vital in solving complicated problems of physics and engineering- from pendulum motion, electrical circuit design, to magnetic field calculations in elliptical coordinates.

More particularly, Gaussian quadrature and Romberg's integration are a product of the development of numerical methods, and their corresponding relationship has tremendously improved accuracy and applicability in the evaluation of elliptic integrals. The arithmetic-geometric mean method is an effective method of calculating full elliptic integrals of both the first and second kinds based on its strong convergence properties.

With more of Jacobi's elliptic functions and introduction from the Weierstrass elliptic functions, it was vastly enriched in this area, offering powerful tools for inverting elliptic integrals, powerful tools for the complex analysis unification process.

In a sense, elliptic integrals form a critical intersection of theoretical mathematics with practical application. Research into properties and continuous innovation in computational techniques keep elliptic integrals as a dynamic area at the very center of the mathematical research arena.

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