The Circle Method of Asymptotic Approximation

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History of the Circle Method

In 1916, G. H. Hardy and Ramanujan developed the circle method. They used it to asymptotically approximate the partition function and published a paper on the topic in 1918.

What is the Circle Method?

The circle method is a process of converting a sequence into an integral, and by bounding this integral, finding an asymptotic approximation for the function.

A problem

Let p_n be the number of unordered partitions of n. We have $p_4 = 5$, $p_5 = 7$.

Asymptotic Approximation of the Partition Function

What is the behavior of p_n as n goes to infinity? Stated differently, find a "nice" function f so $\lim_{n\to\infty} \frac{p_n}{f(n)} = 1$.

Generating Functions

Suppose a_n is a sequence. Then we can create the *generating function* $g(x) = \sum_{n=0}^{\infty} a_n x^n$.

Multiplication of Generating Functions

Let $d(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$. What does $d(x)^2$ represent?

Partition Generating Function

Lemma (Formal Product of the Generating Function)

$$1 + \sum_{n=1}^{\infty} p_n z^n = (1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)\cdots$$
$$= \prod_{n=1}^{\infty} \frac{1}{1 - z^n}.$$

Contour Integrals

Consider a parameterization of a curve Γ, $c(t) = u(t) + iv(t)$ for $t \in [0,1]$ (or some arbitrary interval). Then we define

$$\int_\Gamma f(z)\,dz := \int_0^1 f(c(t))\,dt.$$

Cauchy's Residue Theorem

A Special Case of Residue Theorem

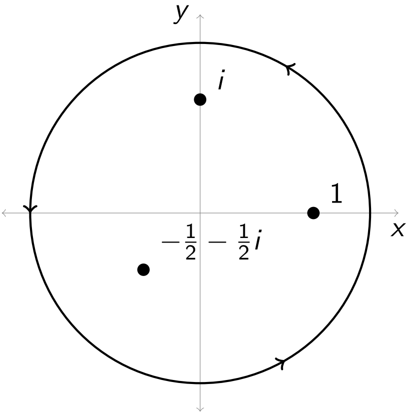
Let C be a circle parameterized counterclockwise, and let P be a set of all poles of f inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{p \in P} \operatorname{Res}(f, p).$$

What is a Residue?

$$f(z) = \cdots + \frac{c_{-2}}{(z-a)^2} + \frac{\mathbf{c_{-1}}}{(z-a)} + c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

Visualization of Residue Theorem



Application to a Common Function

As a corollary of this, we have

$$\int_{C_r} \frac{1}{z} = 2\pi i$$

where C_r is a circle of radius r parameterized counterclockwise.

The Resulting Integral

Lemma

If g is the generating function for a_n,

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{g(z)}{z^{n+1}} \, dz.$$

It is this integral we bound during the circle method.

Major and Minor Arcs

The major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ are defined so

$$\int_{\mathfrak{M}} \frac{g(z)}{z^{n+1}}\, dz = O\left(\int_{\mathfrak{m}} \frac{g(z)}{z^{n+1}}\, dz\right)$$

The Asymptotic Formula

In 1917, G. H. Hardy and Ramanujan discovered an asymptotic formula for the partition function.

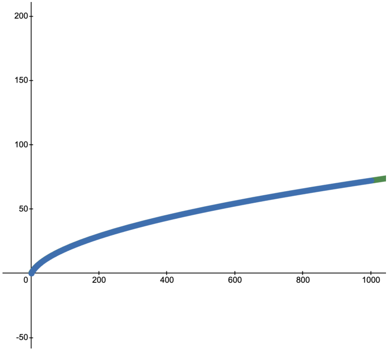
Theorem

Asymptotic Behavior of the Partition Function

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}. \tag{1}$$

Error of the approximation

We graph of the logarithms of the asymptotic behavior and the actual values up to $p(1000)$.



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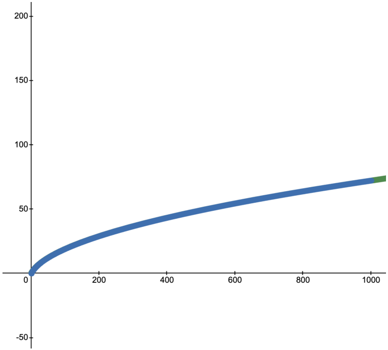

The approximation is actually quite good!

Other Problems

Other problems this have been used for include:

- Waring's problem (by Vinogradov)
- The weak Goldbach conjecture (by Helfgott)

Thank You For Your Attention!