The Circle Method and Approximation of the Partition Function

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Abstract. The Circle Method was created in 1916 by G. H. Hardy and Ramanujan to find the behavior of the Partition function. It was later used to solve the asymptotic form of Waring's problem on the sum of powers, and has been used in efforts to prove the weak Goldbach conjecture. The Circle Method can be summarized in a few steps:

1. Find the generating function of a sequence, a_n.
2. Turn this generating function into an integral.
3. Choose major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.
4. Bound the generating function over these arcs, so the minor arcs contribute less than the major arcs.
5. Use the major arcs to find an asymptotic formula for a_n.

In this paper, we will present a proof that p_n, the number of partitions of n, satisfies

Theorem 1.

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \tag{1}$$

in a proof due to [New62] found in [A'C17].

0.1 Big-O Notation

Definition 1. *We will write $f(x) = O(g(x))$ if and only if there exist $M, x_0 \in \mathbb{R}$ such that*

$$|f(x)| \leq Mg(x)$$

for all $x \geq x_0$.

1 Partitions

Consider p_n, the number of ways to form an unordered partition of n. For example, $p_4 = 5$ because $4 = 1+3 = 2+2 = 1+1+2 = 1+1+1+1$. We would like to determine the behavior of p_n.

1.1 The Formal Product

Let us define $f(z) := 1 + \sum_{k=1}^{\infty} p_k x^k$. To put f into a nice form, we consider what a partition is made of. Each partition of n is of the form $a_1 \cdot 1 + a_2 \cdot 2 + \cdots$, where a_k is the number of k's in the partition. We therefore define

$$P(z) := (1 + z^1 + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)(1 + z^3 + z^6 + \cdots)\cdots$$

To get a partition of n, we choose z^{a_1} from the first sum, z^{2a_2} from the second sum, and so on. Manipulating this product formally, we have

$$\begin{aligned} P(z) &= \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} z^{mn} \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - z^n} \end{aligned}$$

if $|z| < 1$. Now, we would like to work with $P(z)$ rather than $f(z)$; however, we must make sure it converges first.

Lemma 2. *When $|z| < 1$, the product $P(z)$ converges absolutely, thus $f(z) = P(z)$.*

Proof. Consider the series

$$\log P(z) = -\sum_{n=1}^{\infty} \log(1 - z^n).$$

If we know this series converges, we know the product must converge because exp is continuous. Since $|z| < 1$, the Taylor series of log converges, thus we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} |\log(1 - z^n)| &= \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{z^{mn}}{m} \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \frac{z^{mn}}{m} \right| \\
&\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |z^{mn}| \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |z|^{mn} \\
&= \sum_{n=1}^{\infty} \frac{|z|^n}{1 - |z|^n} \\
&\leq \sum_{n=1}^{\infty} \frac{|z|^n}{1 - |z|} \\
&< \sum_{n=1}^{\infty} \frac{|z|^n}{1 - |z|} \\
&= \frac{|z|}{(1 - |z|)^2}.
\end{aligned}
$$

Therefore $P(z)$ is absolutely convergent. Assuming $|z| < 1$, we must have $f(z) = P(z)$ by rearranging the terms. □

1.2 Approximating the Generating Function

Normally when applying the circle method, we would like to express p_n in the form

$$p_n = \oint_{C_r} \frac{f(z)}{z^{n+1}} \, dz.$$

However, here we will instead find an approximation of f and use the circle method on their difference. When we do this, we expect the difference to be at its largest near 1 because the formal product has many factors of $\frac{1}{1-z}$. In fact, an arc around 1 will be our only major arc.

To make this approximation, we want to consider a function that will be continuous when z is near 1, as that will be easier to work with. We consider the logarithm of the formal product and remove the pole via a few terms. The reader can consider the Taylor series of the functions to see the order of these poles. In this, we set $z = e^{-w}$ as it makes the substitution nice.

Since the sum in $\log f$ absolutely converges, we can switch the order of the sum and

$$\log f(e^{-w}) = \sum_{n=1}^{\infty} \frac{e^{-nw}}{n(1 - e^{-nw})} = w \sum_{n=1}^{\infty} \frac{1}{nw(e^{nw} - 1)}. \tag{2}$$

Now, we consider what an approximation of $\frac{1}{u(e^u-1)}$ would be. Since we are observing its behavior around $u = 0$ as w is close to 0, we want to subtract the poles and consider the difference later. Let us analyze $\frac{1}{u(e^u-1)}$ around 0. Using the Taylor expansion of $e^u - 1$, we can see that

$$\begin{aligned} \frac{1}{u(e^u - 1)} &= \frac{1}{u(-1 + 1 + u + \frac{u^2}{2} + \cdots)} \\ &= \frac{1}{u(u + \frac{u^2}{2} + \cdots)} \\ &= \frac{1}{u^2(1 + \frac{u}{2} + \cdots)}. \end{aligned}$$

Thus we would expect $\frac{1}{u(e^u-1)}$ to have a pole of maximal order 2. In fact,

$$\begin{aligned} \lim_{u \to 0} \frac{u^2}{u(e^u - 1)} &= \lim_{u \to 0} \frac{u}{e^u - 1} \\ &= \lim_{u \to 0} \frac{1}{e^u} \\ &= 1, \end{aligned}$$

by L'Hôpital's rule. We will then subtract $\frac{1}{u^2}$ to remove this pole. When considering this in the infinite sum of Equation 2, we see that it will result in a term of $\frac{\pi^2}{6}$. We then calculate the residue of the first order pole,

$$\begin{aligned} \lim_{u \to 0} u \left(\frac{1}{u(e^u - 1)} - \frac{1}{u^2} \right) &= \lim_{u \to 0} \frac{1}{e^u - 1} - \frac{1}{u} \\ &= \lim_{u \to 0} \frac{u - e^u + 1}{u(e^u - 1)} \\ &= \lim_{u \to 0} \frac{1 - e^u}{e^u - 1 + ue^u} \\ &= \lim_{u \to 0} \frac{-e^u}{e^u + e^u + ue^u} \\ &= -\frac{1}{2}. \end{aligned}$$

In this case, just adding $\frac{1}{2u}$ will not work, because in the sum of Equation 2 we would have a sum of $\sum_{n=1}^{\infty} \frac{1}{2u} = \infty$. Therefore we use $\frac{e^{-u}}{2u}$ because summing this results in a logarithm via the Taylor series and its $\frac{1}{u}$ term has the right coefficient. If we call our error term $g(u)$, we have

$$g(u) := \frac{1}{u(e^u - 1)} - \frac{1}{u^2} + \frac{e^{-u}}{2u} \tag{3}$$

and

$$\log f(e^{-w}) = w \sum_{n=1}^{\infty} \left(g(nw) + \frac{1}{(nw)^2} - \frac{e^{-nw}}{2nw} \right)$$
$$= w \sum_{n=1}^{\infty} g(nw) + \frac{1}{w} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-nw}}{n} \tag{4}$$
$$= w \sum_{n=1}^{\infty} g(nw) + \frac{\pi^2}{6w} + \frac{1}{2} \log(1 - e^{-nw}).$$

We have a sum which is hard to work with, so we will convert it into an integral and bound the difference. Now, we state a technical theorem:

Theorem 3.

$$\int_0^{\infty} g(u)\, du = -\frac{1}{2} \log 2\pi.$$

Proof. The proof of this theorem is too long to include in this paper. It is in [A'C17], Pages 9-11. However, this theorem is useful because of the next section. □

1.3 The Total Variation

We begin this section with a definition.

Definition 2. *The total variation of γ, denoted V_γ, is defined as*

$$V_\gamma = \sup \left\{ \sum_{j=0}^{k-1} |\gamma(t_{j+1}) - \gamma(t_j)| : k \in \mathbb{N}, 0 \leq t_0 < t_1 < \cdots < t_k \right\}.$$

Why will we use the total variation? Because it bounds exactly what we want, in fact,

Lemma 4. *If $\gamma : [0, \infty) \to \mathbb{C}$ is continuous and integrable*

$$\left| \int_0^{\infty} \gamma(t)\, dt - \sum_{n=1}^{\infty} \gamma(n) \right| \leq 2V_\gamma$$

Proof. We turn the integral on $(0, \infty)$ into infinitely many integrals on unit intervals:

$$\int_0^{\infty} \gamma(t)\, dt = \sum_{n=0}^{\infty} \int_n^{n+1} \gamma(t)\, dt.$$

Suppose $\gamma(t) = u(t) + iv(t)$. We know u and v are continuous, therefore we use the Mean Value Theorem for integrals:

$$\int_n^{n+1} \gamma(t)\, dt = u(a_n) + iv(b_n)$$

for some $a_n, b_n \in [n, n+1]$. Therefore

$$\begin{aligned}\left|\int_0^\infty \gamma(t)\,dt - \sum_{n=1}^\infty \gamma(n)\right| &= \left|\sum_{n=0}^\infty (u(a_n) + iv(b_n)) - \sum_{n=0}^\infty (u(n+1) + iv(n+1))\right| \\ &\le \sum_{n=0}^\infty |u(a_n) - u(n+1)| + \sum_{n=0}^\infty |v(a_n) - v(n+1)| \\ &\le \sum_{n=0}^\infty |\gamma(a_n) - \gamma(n+1)| + \sum_{n=0}^\infty |\gamma(a_n) - \gamma(n+1)|\,.\end{aligned}$$

Set $\alpha_{2n} = a_n$, $\alpha_{2n+1} = n+1$, $\beta_{2n} = b_n$, and $\beta_{2n+1} = n+1$. Each sum on the last line is a variation over one of the sequences but missing terms. Therefore

$$\begin{aligned}\left|\int_0^\infty \gamma(t)\,dt - \sum_{n=1}^\infty \gamma(n)\right| &\le \sum_{n=0}^\infty |\gamma(\alpha_n) - \gamma(\alpha_{n+1})| + \sum_{n=0}^\infty |\gamma(\beta_n) - \gamma(\beta_{n+1})| \\ &\le 2V_\gamma.\end{aligned}$$

□

Now we find a bound for V_γ that can be more easily used:

Lemma 5. *If γ, in addition to being continuous and integrable, is differentiable, we have*

$$V_\gamma \le \int_0^\infty |\gamma'(t)|\,dt.$$

Proof. We know

$$\begin{aligned}\sum_{j=0}^{k-1} |\gamma(t_{j+1}) - \gamma(t_j)| &= \sum_{j=0}^{k-1} \left|\int_{t_j}^{t_{j+1}} \gamma'(t)\right| dt \\ &\le \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} |\gamma'(t)|\ dt \\ &= \int_{t_0}^{t_k} |\gamma'(t)|\ dt \\ &\le \int_0^\infty |\gamma'(t)|\ dt.\end{aligned}$$

Therefore the supremum also satisfies this inequality. □

1.4 Using the Total Variation

We would like to apply the last section to $\gamma : t \mapsto g(wt)$. However, first we must make sure g is holomorphic (complex differentiable in a neighborhood of every point in the domain) on the line parameterized by wt for $t \in [0, \infty)$. We know g has a pole when $e^u - 1 = 0$ and $u \neq 0$, or $u = 2\pi ik$ where $k \neq 0$ because we removed the pole at 0. Since we are interested in z inside the unit circle, we are only interested in $-w$ with real part strictly negative, so $|z| = |e^{-w}| = |e^{\Re(-w)}| < 1$. Therefore, we do not need to consider $\Re(u) \leq 0$ where we will be substituting w. We know g is continuous on this half-plane.

We now consider g'. We have

$$g'(u) = \frac{2}{u^3} - \frac{e^{-u}}{2u^2} - \frac{1}{u^2\left(e^u - 1\right)} - \frac{e^{-u}}{2u} - \frac{e^u}{u\left(e^u - 1\right)^2} \tag{5}$$

from calculation; we have already removed the pole so this will g has continuous derivative with $|u| < 1$. For larger $|u|$ we factor the derivative:

$$g'(u) = \frac{1}{u^2}\left(\frac{2}{u} - \frac{1}{2e^u} - \frac{1}{e^u - 1} - \frac{u}{2e^u} - \frac{ue^u}{\left(e^u - 1\right)^2}\right).$$

On a line parameterized by wt, the e^{wt} dominates, thus all the terms in the parentheses go to 0, and $u^2 g'(u) \to 0$ as $|u| \to \infty$. We will make this convergence absolute so we can take the integral. Since we must bound e^{-u}, we pick $0 < K < \pi$ and restrict the domain of g to $T = \{z \in \mathbb{C} \mid |\arg(z)| < K\}$. Then $|e^u| = e^{\Re(u)} = e^{|u| \cos \arg u} \geq e^{|u| \cos(K)}$. Then all the terms each go to zero; the last one does because $\frac{ue^u}{(e^u-1)^2} \sim \frac{ue^u}{e^{2u}} = \frac{u}{e^u}$. This means we must have some $M(K)$ so $u^2 g(u) \leq M(K)$ for $|u| > 1$. We will assume for now that we have chosen some K.

We then have

$$\left|\sum_{n=1}^{\infty} g(nw) - \int_0^{\infty} g(tw)\, dt\right| \leq 2V$$

with V defined as the total variation on $g(tw)$ with t as the input. But we know

$$\begin{aligned}\left|\sum_{n=1}^{\infty} g(nw) - \int_0^{\infty} g(tw)\, dt\right| &= \left|\sum_{n=1}^{\infty} g(nw) - \int_0^{\infty} g(tw)\, dt\right| \\ &= \left|\sum_{n=1}^{\infty} g(nw) - \frac{1}{w}\int_{L_w} g(u)\, du\right| \\ &= \frac{1}{w}\left|w\sum_{n=1}^{\infty} g(nw) - \int_{L_w} g(u)\, du\right|.\end{aligned}$$

Now, we would like to compare the integral along L_w and the integral from 0 to ∞. To do this, we create a sequence of closed contours that are a counterclockwise parameterization of the perimeter of a sector of a circle defined by $\sigma + 0i$ and L_w with some radius r, an example arc is shown in the following picture.

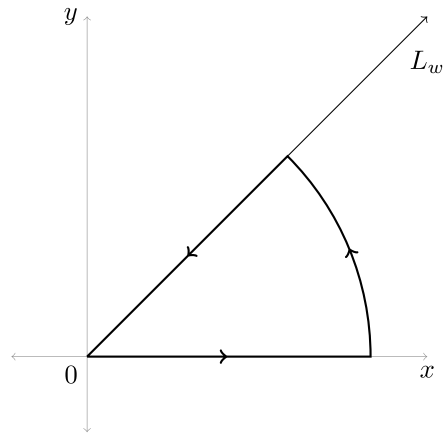

Figure 1: Contour integral on g

We have already found the complex poles, and know they are not in this sector. Thus we only have to consider the arc. We see that in the definition of g in Equation 3, we know g is $O\left(\frac{1}{u^2}\right)$. Thus, even with the expanding length of the arc, the contour integral along this arc goes to 0 as r goes to ∞. This means that the integral from 0 to ∞ plus the integral from ∞ to 0 on L_w is 0 by Residue Theorem and

$$\int_0^\infty g(t)\,dt = \int_{L_w} g(z)\,dz.$$

We thus have

$$\frac{1}{|w|}\left|w\sum_{n=1}^{\infty} g(nw) - \int_0^\infty g(t)\,dt\right| \leq 2V$$

and

$$\left|w\sum_{n=1}^{\infty} g(nw) - \int_0^\infty g(t)\,dt\right| \leq 2|w|V.$$

Now, we have calculated that $g'(wt) = O\left(\frac{1}{w^2t^2}\right)$. Then

$$\begin{aligned}\int_{L_w} g'(u)\,du &= w\int_0^\infty g'(wt)\,dt \\ &= O\left(w\frac{1}{wt}\right)\end{aligned}$$

Rearranging, using Theorem 3, and accounting for the extra $|w|$ above, we have

$$w\sum_{n=1}^{\infty} g(nw) = -\frac{1}{2}\log 2\pi + O(w).$$

Now, returning to Equation 4, we have

$$\log f(e^{-w}) = \frac{\pi^2}{6w} + \frac{1}{2}\log(1 - e^{-w}) - \frac{1}{2}\log 2\pi + O(w).$$

We would like to turn this into a function of z, since we want an approximation of $f(z)$. We have

$$w = -\log z = -\log(1 - (1 - z)) = (1 - z) + \frac{(1 - z)^2}{2} + \cdots .$$

When z is close to 1, $w = O(1 - z)$. We also know

$$\begin{aligned}
\frac{1}{w} &= \frac{1}{(1 - z) + \frac{(1-z)^2}{2} + \cdots} \\
&= \frac{1}{1 - z} + \frac{1}{(1 - z)(1 + \frac{(1-z)}{2} + \frac{(1-z)^2}{3} \cdots)} - \frac{1}{1 - z} \\
&= \frac{1}{1 - z} + \frac{1 - (1 + \frac{(1-z)}{2} + \frac{(1-z)^2}{3} \cdots)}{(1 - z)(1 + \frac{(1-z)}{2} + \frac{(1-z)^3}{3} \cdots)} \\
&= \frac{1}{1 - z} + \frac{-\frac{1}{2} - \frac{(1-z)}{3} \cdots}{(1 - z)(1 + \frac{(1-z)}{2} + \frac{(1-z)^3}{3} \cdots)} \\
&= \frac{1}{1 - z} + \frac{-\frac{1}{2} - \frac{(1-z)}{3} \cdots}{1 + \frac{(1-z)}{2} + \frac{(1-z)^3}{3} \cdots} \\
&= \frac{1}{1 - z} - \frac{1}{2} + O(1 - z).
\end{aligned}$$

We now have

$$\begin{aligned}
\log f(z) &= \frac{\pi^2}{6}\left(\frac{1}{1 - z} - \frac{1}{2} + O(1 - z)\right) + \frac{1}{2}\log\left(\frac{1 - z}{2\pi}\right) + O(1 - z) \\
&= \frac{\pi^2}{6(1 - z)} - \frac{\pi^2}{12} + \frac{1}{2}\log\left(\frac{1 - z}{2\pi}\right) + O(1 - z).
\end{aligned}$$

Therefore, let us set ϕ to be the non-error term of this equation, i.e.

$$\phi(z) := \sqrt{\frac{1 - z}{2\pi}}\exp\left(\frac{\pi^2}{6(1 - z)} - \frac{\pi^2}{12}\right),$$

so

$$\log f(z) = \log\phi(z) + O(1 - z)$$

and

$$\begin{aligned}
f(z) &= \phi(z)e^{O(1-z)} \\
&= \phi(z)(1 + O(1 - z) + \frac{1}{2}O\left((1 - z)^2\right) + \dots).
\end{aligned}$$

Therefore

$$f(z) = \phi(z)(1 + O(1 - z)). \tag{6}$$

1.5 Defining the Major and Minor Arcs

If the coefficients of f and ϕ are p_n and q_n, by Residue Theorem, we have

$$p_n - q_n = \int_{C_r} \frac{|f(z) - \phi(z)|}{z^{n+1}} \, dz.$$

Now, motivated by the pole at 1, we will choose our major and minor arcs. We choose the radius of our circle, C, to be $1 - v(n)$, and our major arc to be near 1, so we define $\mathfrak{M} = \{z \in C : |z - 1| < s(n)\}$ where we choose some $s(n), v(n) \to 0$ as $n \to \infty$.

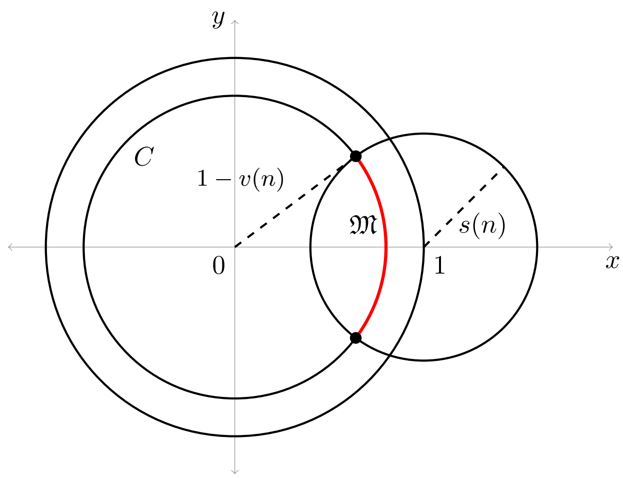

Figure 2: The Major Arc

To choose $s(n), v(n)$, and ultimately choose K, we state the following lemma.

Lemma 6. *Suppose $w \in \mathbb{C}$ with $|\arg w| < \pi$ and $e^{-w} \in \mathfrak{M}$. For large enough n,*

$$|\arg w| \le \arccos \frac{v(n)}{2s(n)}.$$

Proof. Since $e^{-w} \in C$, we have

$$e^{\Re(-w)} = |e^{-w}| = 1 - v(n).$$

Since $\exp(-x)$ is convex, $\exp(0) = 1$, and $\exp'(0) = 1$, $1 - \Re(w) \le e^{-\Re(w)}$. We therefore have $v(n) \le \Re(w)$. Now, for sufficiently large n, w is close to 0 since $e^{-w} \in \mathfrak{M}$. Then

$$\frac{1}{2} \le \frac{|e^{-w} - 1|}{|w|}.$$

By definition, $|e^{-w} - 1| < s(n)$, so

$$\frac{1}{2} \leq \frac{s(n)}{|w|} = \frac{s(n)}{v(n)|w|} v(n) \leq \frac{s(n)}{v(n)} \frac{\Re(w)}{|w|} = \frac{s(n)}{v(n)} \cos \arg w.$$

Thus

$$\cos \arg w \geq \frac{2v(n)}{s(n)}$$

□

We have a bound for $\arg w$; thus we shall make this bound easy to work with. Setting $s(n) = c_s n^{-t}$ and $v(n) = c_v n^{-t}$ for some c_s, c_v, t yields constant $K = \arccos \frac{2v(n)}{s(n)} = \arccos \frac{2c_v}{c_s}$. Therefore we can use the approximation in Equation 6.

1.6 Bounding the Major Arc

We have

$$\begin{aligned} \int_{\mathfrak{M}} \frac{f(z) - \phi(z)}{z^{n+1}} \, dz &= \int_{\mathfrak{M}} \frac{(1 + O(1-z))\phi(z) - \phi(z)}{z^{n+1}} \, dz \\ &= \int_{\mathfrak{M}} z^{-n-1} O(1-z)\phi(z) \, dz \\ &= \int_{\mathfrak{M}} O(z^{-n-1}(1-z)\phi(z)) \, dz \\ &= O\left(\int_{\mathfrak{M}} z^{-n-1}(1-z)\phi(z) \, dz \right). \end{aligned} \tag{7}$$

To bound $|z|^n$, we use the next lemma.

Lemma 7. *We have*

$$|z|^n = O\left(e^{c_v n^{1-t}} \right).$$

Proof. We do this by proving $\lim_{n\to\infty} \frac{(1-an^{-t})^n}{e^{an^{1-t}}} = 1$. We take the logarithm of the numerator and denominator and compare them. Using L'Hôpital's Rule, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{-n \log(1 - an^{-t})}{an^{1-t}} &= \lim_{n\to\infty} \frac{-\log(1 - an^{-t})}{an^{-t}} \\ &= \lim_{n\to\infty} \frac{1}{1 - an^{-t}} \cdot \frac{-tan^{-t-1}}{-tan^{-t-1}} \\ &= 1. \end{aligned}$$

Then we have

$$\begin{aligned} |z|^n &= (1 - v(n))^{-n} \\ &= (1 - c_v n^{-t})^n \\ &= O(e^{c_v n^{1-t}}). \end{aligned}$$

□

Now, we substitute this into Equation 7:

$$\int_{\mathfrak{M}} \frac{f(z)-\phi(z)}{z^{n+1}}\,dz = O\left(\int_{\mathfrak{M}} z^{-n-1}(1-z)\phi(z)\,dz\right)$$
$$= O\left(\int_{\mathfrak{M}} z^{-1}(1-z)\exp\left(c_v n^{1-t}\right)\sqrt{\frac{1-z}{2\pi}}\exp\left(\frac{\pi^2}{6(1-z)}-\frac{\pi^2}{12}\right)\,dz\right)$$
$$= O\left(\int_{\mathfrak{M}} |1-z|^{\frac{3}{2}}\exp\left(c_v n^{1-t}+\frac{\pi^2}{6(1-z)}\right)\,dz\right).$$

Here we drop terms that make the equation simpler yet do not violate the definition of 1 ($z^{-1}\approx 1$)[1]. Now, using our definition of the major arc, where $|1-z|<s(n)=c_s n^{-t}$ and $|1-z|^{-1}\le(1-|z|)^{-1}=v(n)^{-1}=c_v^{-1}n^t$, we have

$$\int_{\mathfrak{M}} \frac{f(z)-\phi(z)}{z^{n+1}}\,dz = O\left(\int_{\mathfrak{M}} \left(c_s n^{-t}\right)^{\frac{3}{2}}\exp\left(c_v n^{1-t}+\frac{\pi^2 n^t}{6c_v}\right)\,dz\right)$$
$$= O\left(\int_{\mathfrak{M}} n^{-\frac{3}{2}t}\exp\left(c_v n^{1-t}+\frac{\pi^2 n^t}{6c_v}\right)\,dz\right).$$

Finally, we also know the length of $\mathfrak{M}$ is $O(n^{-t})$ because the radius of both circles are also $O(n^{-t})$, so

$$\int_{\mathfrak{M}} \frac{f(z)-\phi(z)}{z^{n+1}}\,dz = O\left(n^{-\frac{5}{2}t}\exp\left(c_v n^{1-t}+\frac{\pi^2 n^t}{6c_v}\right)\right).$$

To create the best bound, we see that we have both n^t and n^{1-t}; we thus choose $t=\frac{1}{2}$. To choose c_v we take the parameter of the exponential and look for its minimum with respect to c_v. It is achieved at $c_v=\frac{\pi}{\sqrt{6}}$. Thus we collect the terms and

$$\int_{\mathfrak{M}} \frac{f(z)-\phi(z)}{z^{n+1}}\,dz = O\left(n^{-\frac{5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right). \tag{8}$$

1.7 Bounding the Minor Arc

We will prove another bound on f for this case.

Lemma 8. *For* $|z|<1$,

$$|\log f(z)| < \frac{1}{|1-z|}+\frac{1}{1-|z|}.$$

Proof. We switch the order of the sum of $\log f$ because it is absolutely conver-

[1]Though we could eliminate this via the integral, we do not here because our bound works with the minor arc case well, too.

gent; thus

$$\begin{aligned}\log f(z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{z^{mn}}{m}\\ &= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}z^{mn}\\ &= \sum_{m=1}^{\infty}\frac{z^m}{m(1-z^m)}.\end{aligned}$$

Then

$$\begin{aligned}|\log f(z)| &\le \sum_{m=1}^{\infty}\left|\frac{z^m}{m(1-z^m)}\right|\\ &= \left|\frac{z}{1-z}\right| + \sum_{m=2}^{\infty}\frac{|z|^m}{m(1-|z|^m)}\\ &= \left|\frac{z}{1-z}\right| + \frac{1-|z|}{1-|z|}\sum_{m=2}^{\infty}\frac{|z|^m}{m(1-|z|^m)}\\ &= \left|\frac{z}{1-z}\right| + \frac{1}{1-|z|}\sum_{m=2}^{\infty}\frac{|z|^m}{m(1+|z|+\cdots+|z|^{m-1})}\\ &\le \left|\frac{1}{1-z}\right| + \frac{1}{1-|z|}\sum_{m=2}^{\infty}\frac{1}{m(|z|^{-1}+\cdots+|z|^{-m})}.\end{aligned}$$

Since $|z|<1$, the second sum is less than $\sum_{m=2}^{\infty}\frac{1}{m^2}$. This sum is less than 1, and thus the lemma is true. □

We now have

$$\begin{aligned}\left|\int_{\mathfrak{m}}\frac{f(z)-\phi(z)}{z^{n+1}}\,dz\right| &\le \int_{\mathfrak{m}}\frac{|f(z)|+|\phi(z)|}{|z|^{n+1}}\,dz\\ &< \int_{\mathfrak{m}}\frac{1}{|z|^{n+1}}\left(\exp\left(\frac{1}{|1-z|}+\frac{1}{1-|z|}\right)+|\phi(z)|\right)dz.\end{aligned}$$

To bound this, we use the definition of the minor arcs: we have $|1-z|\ge\frac{c_s}{\sqrt{n}}$ and $1-|z|=\frac{\pi}{\sqrt{6n}}$ which results in

$$\left|\int_{\mathfrak{m}}\frac{f(z)-\phi(z)}{z^{n+1}}\,dz\right| < \int_{\mathfrak{m}}\frac{1}{|z|^{n+1}}\left(\exp\left(\frac{\sqrt{n}}{c_s}+\frac{\sqrt{6n}}{\pi}\right)+|\phi(z)|\right)dz.$$

For $|\phi(z)|$ we use $|1-z|^{-1}\le(1-|z|)^{-1}=\frac{\sqrt{6n}}{\pi}$. We have

$$\begin{aligned}\phi(z) &= \sqrt{\frac{1-z}{2\pi}}\exp\left(\frac{\pi^2}{6(1-z)}-\frac{\pi^2}{12}\right)\\ &= O\left(\sqrt{1-z}\exp\left(\frac{\pi^2}{6(1-z)}\right)\right)\end{aligned}$$

so

$$\begin{aligned}
|\phi(z)| &= O\left(\sqrt{1-z}\exp\left(\frac{\pi^2}{6(1-z)}\right)\right)\\
&= O\left(\sqrt{1-z}\exp\left(\Re\left[\frac{\pi^2}{6(1-z)}\right]\right)\right)\\
&= O\left(\sqrt{1-z}\exp\left(\left|\frac{\pi^2}{6(1-z)}\right|\right)\right)\\
&= O\left(\sqrt{1-z}\exp\left(\frac{\pi^2}{6}\frac{\sqrt{n}}{c_s}\right)\right)\\
&= O\left(\sqrt{1-z}\exp\left(\frac{\pi^2\sqrt{n}}{6c_s}\right)\right).
\end{aligned}$$

We again have by Lemma 7 that

$$|z|^{-n} = O\left(e^{\pi\sqrt{\frac{n}{6}}}\right).$$

Substituting the bound on ϕ into the integral we have

$$\begin{aligned}
\left|\int_{\mathfrak{m}} \frac{f(z)-\phi(z)}{z^{n+1}}\,dz\right| &= O\left(\int_{\mathfrak{m}} \frac{1}{|z|^{n+1}}\left(\exp\left(\frac{\sqrt{n}}{c_s}+\frac{\sqrt{6n}}{\pi}\right)+\sqrt{1-z}\exp\left(\frac{\pi^2}{6}\frac{\sqrt{n}}{c_s}\right)\right)dz\right)\\
&= O\left(\int_{\mathfrak{m}} \frac{1}{|z|}\left(\exp\left(\pi\sqrt{\frac{n}{6}}+\frac{\sqrt{n}}{c_s}+\frac{\sqrt{6n}}{\pi}\right)+\sqrt{1-z}\exp\left(\pi\sqrt{\frac{n}{6}}+\frac{\pi^2\sqrt{n}}{6c_s}\right)\right)dz\right)\\
&= O\left(\exp\left(\pi\sqrt{\frac{n}{6}}+\frac{\sqrt{n}}{c_s}+\frac{\sqrt{6n}}{\pi}\right)+\exp\left(\pi\sqrt{\frac{n}{6}}+\frac{\pi^2\sqrt{n}}{6c_s}\right)\right).
\end{aligned}$$

We now just choose c_s so

$$\frac{\pi}{\sqrt{6}}+\frac{1}{c_s}+\frac{\sqrt{6}}{\pi} < \pi\sqrt{\frac{2}{3}}$$

and

$$\frac{\pi}{\sqrt{6}}+\frac{\pi^2}{6c_s} < \pi\sqrt{\frac{2}{3}}$$

because as long as the exponential term grows faster, the whole error term on the major arc will grow faster. A sufficient example of some c_s that satisfies both of these equations is $c_s = \frac{2\pi}{\sqrt{6}}$. Thus by both of these estimates we have

$$p_n = q_n + O\left(n^{-\frac{5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right). \tag{9}$$

1.8 Asymptotic Behavior of q_n

We turn ϕ into an integral to eliminate the square root.

Lemma 9. *For $a, b \in \mathbb{R}$ with $a > 0$,*

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = \sqrt{\frac{\pi}{a}} \cdot e^{\frac{b^2}{4a}}.$$

Proof. We complete the square in the integral:

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{b}{2a}\right)^2}\,dx.$$

We now set $u = x + \frac{b}{2a}$ so the integral is equal to

$$\int_{-\infty}^{\infty} e^{-au^2}\,du.$$

Finally, we set $v = \sqrt{a}u$ so $dv = \sqrt{a}du$. The integral is thus

$$\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2}\,dv = \sqrt{\frac{\pi}{a}}.$$

The lemma follows by substituting this integral into the previous equation. □

We now rearrange to

$$\begin{aligned}\phi(z) &= \sqrt{\frac{1-z}{2\pi}} \exp\left(\frac{\pi^2}{6(1-z)} - \frac{\pi^2}{12}\right) \\ &= \sqrt{\frac{\pi}{1-z}} \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} (1-z) \exp\left(\frac{\frac{\pi^2}{6}}{1-z}\right).\end{aligned}$$

We now set $a = 1-z$ and $b = -\pi\sqrt{\frac{2}{3}}$. Then

$$\phi(z) = \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} (1-z) \int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{\frac{2}{3}}x}\,dx$$

We write this integral using a power series:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(1-z)x^2+\pi\sqrt{\frac{2}{3}}x}\,dx &= \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} e^{-z^2}\,dx \\ &= \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} z^n\,dx.\end{aligned}$$

We can flip the summation and integral since the power series converges absolutely. Thus the integral is

$$\sum_{n=0}^{\infty} z^n \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \frac{x^{2n}}{n!}\,dx.$$

We now have

$$
\begin{aligned}
q_n &= \int_C \frac{\phi(z)}{z^{n+1}}\, dz \\
&= \int_C z^{-(n+1)} \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} (1-z) \sum_{k=0}^{\infty} z^k \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \frac{x^{2k}}{k!}\, dx\, dz \\
&= \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \int_C \sum_{k=0}^{\infty} \left(z^{k-n-1} - z^{k-n}\right) \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \frac{x^{2k}}{k!}\, dx\, dz \\
&= \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \int_C \sum_{k=0}^{\infty} z^{k-n-1} \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \left(\frac{x^{2k}}{k!} - \frac{x^{2k-2}}{k!}\right) dx\, dz,
\end{aligned}
$$

where we collect the z^{k-n-1} terms. By Cauchy's Residue Theorem, we only need to consider the z^{-1} term, so

$$
q_n = \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2+\pi\sqrt{\frac{2}{3}}x} \left(\frac{x^{2n}}{n!} - \frac{x^{2n-2}}{n!}\right) dx
$$

We substitute $x \to x + \sqrt{n}$, which, ironically, makes the expression work well.

$$
\begin{aligned}
q_n &= \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}n!} \int_{-\infty}^{\infty} (x+\sqrt{n})^{2n-2} e^{-(x^2+2\sqrt{n}x+n)+\pi\sqrt{\frac{2}{3}}x+\pi\sqrt{\frac{2n}{3}}} \left(x^2 + 2\sqrt{n}x\right) dx \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \frac{e^n}{n!} \int_{-\infty}^{\infty} (x+\sqrt{n})^{2n-2} \left(x^2 + 2\sqrt{n}x\right) e^{-x^2-2\sqrt{n}x+\pi\sqrt{\frac{2}{3}}x}\, dx \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \frac{e^n}{n!} \int_{-\infty}^{\infty} x \left(\sqrt{n}\left(1+\frac{x}{\sqrt{n}}\right)\right)^{2n-2} \left(\sqrt{n}\left(2+\frac{x}{\sqrt{n}}\right)\right) e^{-x^2-2\sqrt{n}x+\pi\sqrt{\frac{2}{3}}x}\, dx \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi\sqrt{2n}} \frac{e^n n^n}{n!} \int_{-\infty}^{\infty} x \left(1+\frac{x}{\sqrt{n}}\right)^{2n-2} \left(2+\frac{x}{\sqrt{n}}\right) e^{-x^2-2\sqrt{n}x+\pi\sqrt{\frac{2}{3}}x}\, dx \\
&= \left(1 + O\left(\frac{1}{n}\right)\right) \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi^{\frac{3}{2}}2n} \int_{-\infty}^{\infty} x \left(1+\frac{x}{\sqrt{n}}\right)^{2n-2} \left(2+\frac{x}{\sqrt{n}}\right) e^{-x^2-2\sqrt{n}x+\pi\sqrt{\frac{2}{3}}x}\, dx
\end{aligned} \tag{10}
$$

by Stirling's approximation formula for $n!$. We now define

$$
s_n(x) := \left(1+\frac{x}{\sqrt{n}}\right)^{2n-2} \left(1+\frac{x}{2\sqrt{n}}\right) e^{-\sqrt{n}x}.
$$

We would like to remove this factor in the integral, thus we state the following lemma:

Lemma 10. *As $n \to \infty$, we have*

$$
\int_{-\infty}^{\infty} x s_n(x) e^{\pi\sqrt{\frac{2}{3}}x-x^2}\, dx = \left(1 + O\left(n^{-\frac{1}{8}}\right)\right) \int_{-\infty}^{\infty} x e^{\pi\sqrt{\frac{2}{3}}x-2x^2}.
$$

Proof. We will prove that their difference is bounded by $O\left(n^{-\frac{1}{8}}\right)$ by bounding the integral

$$\int_{-\infty}^{\infty} n^{\frac{1}{8}} x e^{\pi\sqrt{\frac{2}{3}}x - x^2} \left| s_n(x) - e^{x^2} \right| dx$$

where the bound does not depend on n. We will do this by bounding the integral on certain intervals, namely,

(i) $[-n^{\frac{1}{8}}, n^{\frac{1}{8}}]$

(ii) $[-\sqrt{n}, \infty) \setminus [-n^{-\frac{1}{8}}, n^{-\frac{1}{8}}]$

(iii) $(\infty, -\sqrt{n})$

In case (i) we have

$$\log s_n(x) = (2n-2)\log\left(1 + \frac{x}{\sqrt{n}}\right) + \log\left(1 + \frac{x}{2\sqrt{n}}\right) - 2x\sqrt{n}.$$

We use the Taylor expansion of log. Since $|x| \leq n^{\frac{1}{8}}$, $\frac{x}{\sqrt{n}} = O\left(n^{-\frac{3}{8}}\right)$. Then

$$\begin{aligned}
\log s_n(x) &= (2n-2)\left(1 + \frac{x}{\sqrt{n}} - \frac{x^2}{2n} + O\left(n^{-\frac{9}{8}}\right)\right) + 1 + O\left(n^{-\frac{3}{8}}\right) - 2x\sqrt{n} \\
&= (2n-2)\left(\frac{x}{\sqrt{n}} - \frac{x^2}{2n} + O\left(n^{-\frac{9}{8}}\right)\right) + O\left(n^{-\frac{3}{8}}\right) - 2x\sqrt{n} \\
&= 2x\sqrt{n} - x^2 + O\left(n^{-\frac{1}{8}}\right) + O\left(n^{-\frac{3}{8}}\right) + O\left(n^{-\frac{3}{8}}\right) - 2x\sqrt{n} \\
&= -x^2 + O\left(n^{-\frac{1}{8}}\right).
\end{aligned}$$

Now, we have $s_n(x) = e^{-x^2}\left(1 + O\left(n^{-\frac{1}{8}}\right)\right)$ and

$$n^{\frac{1}{8}}|x|e^{\pi\sqrt{\frac{2}{3}}x - x^2}\left|s_n(x) - e^{-x^2}\right| = O\left(|x|e^{\pi\sqrt{\frac{2}{3}}x - 2x^2}\right).$$

Since the exponential term decays faster than the linear term grows, this is integrable. Now, if $|x| > n^{\frac{1}{8}}$ we can bound the integrand by

$$n^{\frac{1}{8}}|x|e^{\pi\sqrt{\frac{2}{3}}x - x^2}\left|s_n(x) - e^{-x^2}\right| \leq x^2 e^{\pi\sqrt{\frac{2}{3}}x - x^2}\left(|s_n(x)| + e^{-x^2}\right). \tag{11}$$

For case (ii), $x \geq -\sqrt{n}$, so

$$1 + \frac{x}{\sqrt{n}} \leq e^{\frac{x}{\sqrt{n}}}$$

by convexity and

$$\left(1 + \frac{x}{\sqrt{n}}\right)^{2n-2} \leq e^{2x\sqrt{n} - \frac{2x}{\sqrt{n}}}.$$

We then know that

$$\begin{aligned}
|s_n(x)| &\leq e^{2x\sqrt{n}-\frac{2x}{\sqrt{n}}} \left|1+\frac{x}{2\sqrt{n}}\right| e^{-2x\sqrt{n}} \\
&\leq e^{-\frac{2x}{\sqrt{n}}} e^{\frac{x}{2\sqrt{n}}} \\
&= e^{-\frac{3}{2}\frac{x}{\sqrt{n}}} \\
&\leq e^{\frac{3}{2}}.
\end{aligned}$$

We return to Equation 11 and see that the e^{-x^2} term will dominate. Therefore the integral is bounded independent of n in case (ii). Now, in case (iii), we have

$$\begin{aligned}
0 &\leq \left(1+\frac{x}{\sqrt{n}}\right)^2 \\
&= 1+\frac{2x}{\sqrt{n}}+\frac{x^2}{n} \\
&< 1+\frac{2x}{\sqrt{n}}+1+\frac{x^2}{n} \\
&\leq e^{\frac{2x}{\sqrt{n}}+\frac{x^2}{n}}.
\end{aligned}$$

Then

$$\begin{aligned}
|s_n(x)| &\leq \left(1+\left|\frac{x}{2\sqrt{n}}\right|\right) e^{(n-1)\left(\frac{2x}{\sqrt{n}}+\frac{x^2}{n}\right)} e^{-2x\sqrt{n}} \\
&\leq (1+|x|)\, e^{2x\sqrt{n}+x^2-\frac{2x}{\sqrt{n}}-\frac{x^2}{n}} e^{-2x\sqrt{n}} \\
&\leq (1+|x|)\, e^{x^2-2x}.
\end{aligned}$$

Then the integrand in Equation 11 is

$$\begin{aligned}
x^2 e^{\pi\sqrt{\frac{2}{3}}x-x^2}\left(|s_n(x)|+e^{-x^2}\right) &\leq x^2 e^{\pi\sqrt{\frac{2}{3}}x-x^2}\left((1+|x|)\, e^{x^2-2x}+e^{-x^2}\right) \\
&\leq x^2 e^{\pi\sqrt{\frac{2}{3}}x-2x}+|x|^3 e^{\pi\sqrt{\frac{2}{3}}x-2x}+x^2 e^{\pi\sqrt{\frac{2}{3}}x-2x^2}.
\end{aligned}$$

Since $\pi\sqrt{\frac{2}{3}}-2>0$ and $x<0$ for case (iii), the integral is bounded because of exponential decay. □

We now use these bounds. From Equation 10,

$$\begin{aligned}
q_n &= \left(1+O\left(\frac{1}{n}\right)\right) \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi^{\frac{3}{2}}2n} \int_{-\infty}^{\infty} 2x s_n(x) e^{\pi\sqrt{\frac{2}{3}}x-x^2}\, dx \\
&= \left(1+O\left(\frac{1}{n}\right)\right)\left(1+O\left(n^{-\frac{1}{8}}\right)\right) \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi^{\frac{3}{2}}2n} \int_{-\infty}^{\infty} 2x e^{\pi\sqrt{\frac{2}{3}}x-x^2}\, dx \\
&= \left(1+O\left(n^{-\frac{1}{8}}\right)\right) \frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi^{\frac{3}{2}}n} \int_{-\infty}^{\infty} x e^{\pi\sqrt{\frac{2}{3}}x-x^2}\, dx.
\end{aligned}$$

We calculate the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{\pi\sqrt{\frac{2}{3}}x-x^2}\,dx &= \int_{-\infty}^{\infty} -\frac{1}{4}\frac{d}{dx}\left(e^{\pi\sqrt{\frac{2}{3}}x-2x^2}\right)+\frac{\pi}{4}\sqrt{\frac{2}{3}}e^{\pi\sqrt{\frac{2}{3}}x-2x^2}\,dx \\
&= -\frac{1}{4}e^{\pi\sqrt{\frac{2}{3}}x-2x^2}\bigg|_{-\infty}^{\infty}+\frac{\pi}{4}\sqrt{\frac{2}{3}}\int_{-\infty}^{\infty} e^{\pi\sqrt{\frac{2}{3}}x-2x^2}\,dx \\
&= \frac{\pi\sqrt{2}}{4\sqrt{3}}\int_{-\infty}^{\infty} e^{\pi\sqrt{\frac{2}{3}}x-2x^2}\,dx.
\end{aligned}$$

This, by Lemma 9, is

$$\frac{\pi}{4}\sqrt{\frac{2}{3}}\sqrt{\frac{\pi}{2}}e^{\frac{\pi^2}{12}}.$$

We put this together:

$$\begin{aligned}
q_n &= \left(1+O\left(n^{-\frac{1}{8}}\right)\right)\frac{e^{\pi\sqrt{\frac{2n}{3}}-\frac{\pi^2}{12}}}{\pi^{\frac{3}{2}}n}\frac{\pi}{4}\sqrt{\frac{\pi}{3}}e^{\frac{\pi^2}{12}} \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(1+O\left(n^{-\frac{1}{8}}\right)\right).
\end{aligned}$$

Then, since $-1-\frac{1}{8}>-\frac{5}{4}$,

$$\begin{aligned}
p_n &= q_n+O\left(n^{-\frac{5}{4}}e^{\pi\sqrt{\frac{2}{3}}}\right) \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(1+O\left(n^{-\frac{1}{8}}\right)\right)+O\left(n^{-\frac{5}{4}}e^{\pi\sqrt{\frac{2}{3}}}\right) \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}\left(1+O\left(n^{-\frac{1}{8}}\right)\right),
\end{aligned}$$

which is exactly what we wanted.

2 Epilogue

We have just witnessed a long, but simplified version of the proof of the asymptotic approximation of the partition function. Our bounds are not as tight, but we still have the asymptotic formula. The Circle Method is not limited to this problem; it has been used in other problems. The specifics are from [Dru20].

2.1 Waring's Problem

Waring's Problem was posed in 1770. It asks, for fixed k, the smallest s so every integer n can be written in the form $x_1^k+\cdots+x_s^k$. The circle method can solve this problem asymptotically. Hardy and Littlewood defined $G(k)$ to be the smallest s such that the previous condition does not hold for finitely many n. The first bound found by Hardy and Littlewood was

$$G(k) \leq (k-2)2^{k-1} + 5.$$

This was improved over the years and has been constricted to

$$G(k) \leq k(\log k + \log\log k + O(1)).$$

2.2 Goldbach's Conjectures

In 1742, Goldbach posed his famous conjectures. The weak conjecture states that every odd number greater than or equal to 5 can be written as the sum of three primes. Vinogradov proved this for large integers using the circle method. However, when a bound on these large integers was acquired, it was too big to be computed. Vinogradov's student, Borozdkin, found a bound with 4 million digits.

In 1989, J.-R. Chen and T. Wang lowered this bound to $3.33 \cdot 10^{43000}$. M.-C. Liu and T. Wang improved it to $2 \cdot 10^{1436}$ in 2002. This number is still very large. In a preprint in 2012, though, Helfgott lowered the bound to 10^{27}. This paper is accepted by most of the mathematical community and has been accepted for publication by a journal but has yet to be published. Helfgott used the circle method and a sieve method to prove this. The proof is in [Hel15].

2.3 Final Words

While the Circle Method is very powerful, it is also very hard to use. Many problems are very hard, if not impossible, to solve with just the circle method. Because of the nature of poles, the behavior of arcs have to be very tightly bounded. It thus must be used with caution, but may help us with further problems soon.

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