Homomorphism Density

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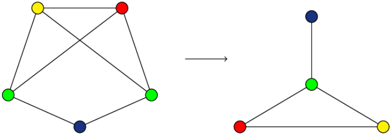
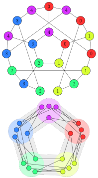
What is a Homomorphism?





A graph homomorphism is an adjacency preserving mapping of vertices between two graphs.
Non-adjacency does not need to be preserved.
We use $\hom(G, H)$ to denote the set of homomorphisms from G to H, and we can write $G \to H$ if there is a homomorphism from G to H.

- Injective Homomorphism: $\hom_{inj}(G, H)$
- Induced Homomorphism: $\hom_{ind}(G, H)$

Homomorphism Density

The density of graph H associated to each graph G is

$$t(H,G) = \frac{|\operatorname{hom}(H,G)|}{n^k}.$$

where $n = V(G)$ and $k = V(H)$.

$$t_{inj}(H,G) = \frac{|\operatorname{hom}_{inj}(H,G)|}{n(n-1)(n-2)\cdot(n-k+1)}.$$

$$t_{ind}(H,G) = \frac{|\operatorname{hom}_{ind}(H,G)|}{n(n-1)(n-2)\cdot(n-k+1)}.$$

Relation between Homomorphisms

$$|\hom_{\text{inj}}(F,G) = \sum_{F' \supseteq F} |\hom_{\text{ind}}(F',G)|$$

$$|\hom(F,G)| = \sum_{P} |\hom_{inj}(F/P,G)|$$

There are also relations for homomorphisms density that can be derived from the relations for homomorphisms.

$$t_{inj}(F,G) = \sum_{F' \supseteq F} t_{ind}(F',G)$$

$$t_{ind}(F,G) = \sum_{F' \supseteq F} (-1)^{e(F')-e(F)} t_{inj}(F',G)$$

Turan's Theorem

Theorem (Turan's Theorem)

If $t(K_r, W) = 0$ *then* $t(K_2, W) \leq (1 - \frac{1}{r-1})$.

Corollary (Mantel's Theorem)

A specific case of Turan's Theorem that was proved before it was $t(K_3, G) = 0$ *then* $t(K_2, G) \leq \frac{1}{2}$.

More Turan's Theorem

Theorem (Goodman's Theorem)

While Mantel's theorem is not inherently interesting, as it is a corollary, but it has been extended to achieve a better bound on the triangle density:

$$t(K_3, G) \geq t(K_2, G)(2 \cdot t(K_2, G - 1)$$

More Goodman's Theorem

Theorem (Kruskal–Katona Theorem)

An upper bound on the density of triangles, which comes from the converse of Goodman's Theorem, which is a corollary to Kruskal–Katona theorem.

$$t(K_3, G) \leq t(K_2, G)^{\frac{3}{2}}$$

Kruskal–Katona Theorem Proof

Proof.

Let $\{\lambda_i\}_{i=1}^{n}$ be the eigenvalues of it's adjacency matrix, A_G. By the definition of a homomophism, $|\hom(K_2, G)| = t(K_2, G)(|V(G)|^2)$. We can convert the homomorphism density to the sum of eigenvalues of A_G,

$$t(K_2, G)(|V(G)|^2) = \sum_{i=1}^{n} \lambda_i^3 \leq \left(\sum_{i=1}^{n} \lambda_i^2 \right)^{\frac{3}{2}} = t(K_2, G)^{\frac{3}{2}}.$$

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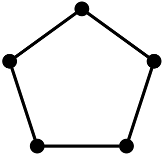
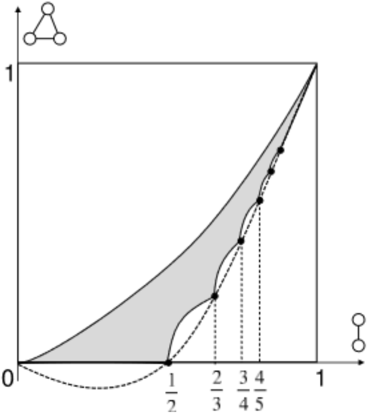


Figure: The 5 node cycle graph

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Visualizing the Results



Useful Connections

Example

$|\hom(G, K_i)|$ is the number of colorings of the graph G with i colors such that no two adjacent nodes have the same color.

Example

If $\hom(K_2, G)$ is k-connected, then the chromatic number of G is at least $k + 3$.

Useful Connections Pt. 2

Example

Given any positive integer k, $|\text{hom}(C_k, G)|$ is the trace of the k-th power of A_G, the adjacency matrix of the graph G.

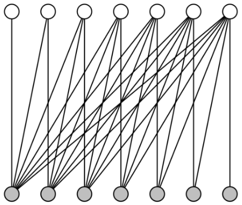
$$|\text{hom}(C_k, G)| = \text{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k,$$

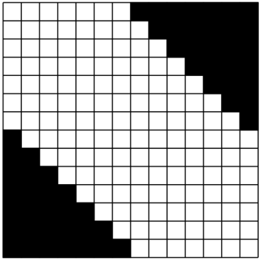
where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A_G.

Example

$$\text{hom}(S_k, G) = \sum_{i \in V(G)} deg(i)^{k-1}$$

Graphons





Definition

A graphon, or graph limit, is a symmetric measurable function W from $[0,1] \times [0,1]$ to $[0,1]$ To compute the graphon, let us find the graphon W_G for a graph G. Label the verticies of G $1 \cdot n$. For a simple graph, partition $[0,1]$ into n equal intervals, each of length $\frac{1}{n}$. For each pair of vertices a and b, let $x \in [\frac{a-1}{n}, \frac{a}{n}]$ and $y \in [\frac{b-1}{n}, \frac{b}{n}]$.

Homomorphisms on Graphons

Definition

We can extend homomorphism density to a graphon W like such:

$$t(H, W) = \int_{[0,1]}^{|V(H)|} \prod_{ij \in E(H)} W(x_i, x_j) \prod_{i \in V(H)} dx_i .$$

Example

For example

$$t(K_3, W) = \int_{[0,1]}^{3} W(x,y)W(y,z)W(z,x)dxdydz$$

.

Graphons X and Y are weakly isomorphic if and only if $t(G, X) = t(G, Y)$ for every graph G.

More on Graphons

We can generalize the Kruskal–Katona theorem to graphons and see

$$D_{2,3} = \{(t(K_2, W), t(K_3, W) : W \text{ is a graphon}\} \in [0,1]^2.$$

Conclusion

Discord: the$_one_star$