Homomorphism Density

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Abstract

This paper will define the homomorphism density of a graph and bound the possible values of it with various bounding theorems. It will further relate graph homomorphisms with the Forbidden Subgraph Problem and offer bounds on the triangle density of graphs. It also will explore connections between homomorphism density and other fields in graph theory before extending homomorphism density to graphons.

1 Introduction

When working with large graphs it is difficult to create quantitative measurements that give meaningful insight on the details of the graph without generating too much information to fully understand the implications. Extremal graph theory helps solve this problem by providing probabilistic insights on the structures of graphs.

A primitive solution is sampling the graph and looking at the probability of properties appearing in a random sample. Homomorphism density is a more sturdy abstraction of this, it is a graph parameter between 0 and 1 that shows us how dense homomorphisms are when compared to another graph. This allows us to measure the similarity between the two graphs with a single parameter. A homormophism is an adjacency preserving mapping between two graphs, which allows some amount of variation before it is no longer considered a homomorphism.

Much of what this paper will focus on is how homomorphism density allows us to understand the relation between the two graphs. For example bounding theorems let us approximate homomorphism densities when given the number of edges in a graph, or limit the number of edges in a graph if a specific p-clique does not appear in it. Triangle density gives us upper and lower bounds on the number of triangles in a graph when given the number of edges and vertices it has or vice-versa using Goodman's Theorem and the Krusal–Katona Theorem. Homomorphism density can be related to other properties of graphs, like the chromatic number or walks, and can be applied to the maxcut and multicut problems.

A secondary part of this paper explores graphons, symmetric measurable functions that represent dense graphs which simplify the counting and identification of homomorphisms. Graphons also allow for extentions of homomorphism theorems and equalities that exist for graphs.

2 Background

As mentioned previously, homomorphism density is a part of extremal graph theory. Extremal graph theory is a field that arises from the combination of extremal combinatorics and graph theory. It deals with how the general parameters of the graph influence local structure. A statement that becomes more apparent when studying homomorphism density. Many problems and theorems in extremal graph theory concern limits and bounds, like Turán's theorem and the Forbidden subgraph problem.

Before we venture any further into homomorphisms, we must first clear a few graph theory concepts out of the way.

To start, a few common graphs are labeled and defined, as they are used throughout this paper. We will almost exclusively be dealing with simple graphs.

Definition 2.1 (Complete Graph)**.** A graph is considered complete if all nodes are connected to each other. Complete graphs are notated with the symbol K. For example K_3 the complete graph with three vertices, which happens to be a triangle.

Definition 2.2 (Cycle Graph)**.** A cycle graph is a graph with contains exactly one cycle, and no more. It is denoted by C_n, where n is the number of vertices.

Definition 2.3 (Star Graph)**.** A star graph, S_k, has a central node, and every other node connects exclusively to the central node.

Now a few common graph theory concepts are defined for the convenience of the reader.

Definition 2.4 (Walk)**.** A walk is a set of edges that join together vertices.

Definition 2.5 (Path)**.** A path is a walk that does not pass the same vertex twice.

Definition 2.6. We consider a graph to be k-vertex-connected (or k-connected for short) it remains connected if we remove any $k-1$ or fewer vertices. For example the complete graph K_n is $n-1$-vertex-connected, and the cyclic graph C_n is 1-vertex-connected.

Definition 2.7 (Complementary Graph)**.** A complementary graph H of G is one where the number of vertices remains the same, but the vertices are adjacent in H if and only if they weren't in G.

Definition 2.8 (Graph Partition)**.** In the context of graph theory, a partition splits vertices of a graph into mutually exclusive sets.

Definition 2.9 (p-clique)**.** A p-clique is a p vertex subset of a graph G such that all the vertices of the subset are adjacent. It can be thought of as a K_p subgraph.

To algebraically manipulate a graph we can represent it as an adjacency matrix. This allows us to calculate eigenvalues and perform other matrix operations on the graph.

Definition 2.10 (Adjacency Matrix)**.** An adjacency matrix is a $n \times n$ matrix used to represent a graph. For a simple graph, the elements in the matrix denote if there is an edge between two nodes.

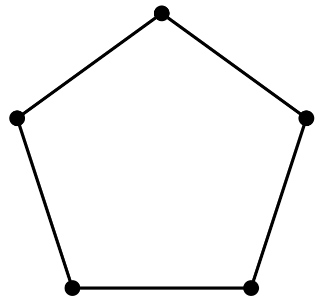

Figure 1. The 5 node cycle graph

For example this is the adjacency matrix for C_5, the 5 node cyclic graph, as seen in figure 1.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This becomes useful when graphons are involved.

3 Graph Homomorphisms

Before we define what Homomorphism Density is, we must first define what a graph homomorphism is.

Definition 3.1. A graph homomorphism is an adjacency-preserving mapping between two graphs. Let $\hom(G, H)$ denote the set of homomorphisms from G to H. $|\hom(G, H)$ is referred to as the homomorphism number. We can write $G \to H$ if there is a homomorphism from G to H. An example is figure 2.

Homomorphisms are a term from algebra that represent a structure preserving mapping. This definition is merely being extended to graphs. These allow us to compare and characterize dense graphs.

While a homomorphism has to preserve adjacency, non-adjacency does not have to be preserved, which means that homomorphisms are not equivalent to subgraphs.

A homomorphism can also apply to graphs with different numbers of vertices, C_6 can map to C_3 trivially by grouping adjacent vertices by two's and mapping each to one vertex of C_3.

Definition 3.2. An injective homomorphisms mapping is, like its name suggests, injective. That is, an injective homomorphism from G to H ensures each vertex of G map to a distinct vertex in H. Let $\hom_{\text{inj}}(G, H)$ denote the set of injective homomorphisms from G to H.

Definition 3.3. An induced homomorphism is a homomorphism that preserves adjacency in addition to non-adjacency. Let $\hom_{\text{ind}}(G, H)$ denote the set of induced homomorphisms from G to H.

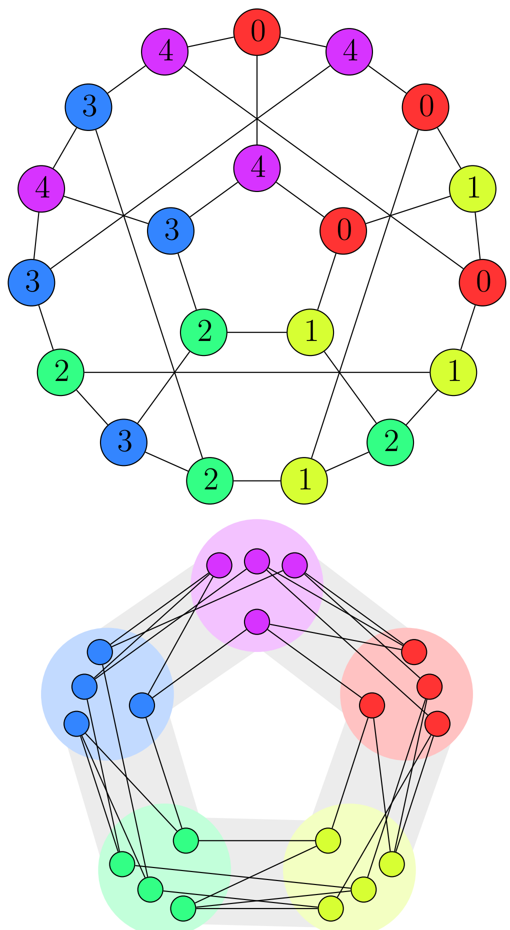

Figure 2. A homomorphism from J_5 to C_5.

Note that $G \to H$ does not imply that G is a subgraph of H, however an injective homomorphism does guarantee this. An induced homomorphism goes further and guarantees that there is a copy of G in H, that is G is an induced subgraph of H. Induced homomorphisms are a subset of injective homomorphisms, which are a subset of general homomorphisms.

Additionally $\hom_{surj}$ may be used to denote a surjective homomorphism mapping, although it is not used in this paper.

Definition 3.4. Let the homomorphism density of a graph H and any other graph G be defined as

$$t(H,G) = \frac{|\hom(H,G)|}{|V(G)|^{|V(H)|}} \tag{3.1}$$

where $V(G)$ is the number of vertices in G and $V(H)$ is the number of vertices in H; an odd normalization that proves to be useful when relating homomorphism density to the rest of extremal graph theory.

As the name suggests the homomorphism density of a graph reaches 1 for more dense graphs and gets close to 0 for sparse graphs.

We use $\mathrm{t_{inj}}$ and $\mathrm{t_{ind}}$ to denote injective and induced homomorphism density respectively. A key difference is that for injective and induced homomorphisms, instead of dividing by $|V(G)|^{|V(H)|}$, we divide by $n(n-1)(n-2)\cdot(n-k+1)$, alternatively expressed as $\frac{n!}{k!}$. This is the case because injective and induced homomorphisms don't measure the probability of a random mapping from F to G being a homomorphism, but rather the probability of a random injective mapping from F to G being a homomorphism.

So we have the two following definitions for injective and induced homomorphism density respectively:

$$\mathrm{t_{inj}}(H,G) = \frac{|\hom_{\mathrm{inj}}(H,G)|}{n(n-1)(n-2)\cdot(n-k+1)} \tag{3.2}$$

and

$$\mathrm{t_{ind}}(H,G) = \frac{|\hom_{\mathrm{ind}}(H,G)|}{n(n-1)(n-2)\cdot(n-k+1)} \tag{3.3}$$

Proposition 3.5. *Given two graphs, G and H, $\mathrm{t_{ind}}(G,H)$ is the probability that a random $V(G)$ vertices of H are the induced graph of G.*

3.1 Relations & Operations

Complementary graphs for induced homomorphism yields a clean equation.

Example. For induced graphs $\mathrm{t_{ind}}(F,\overline{G}) = \mathrm{t_{ind}}(\overline{F},G)$

We can also explore various graph operations:

$$|\hom(F_1 \cup F_2, G)| = |\hom(F_1,G)| \cdot |\hom(F_2,G)| \tag{3.4}$$

if F_1 and F_2 are disjoint.

If F is connected and G is disjoint:

$$|\hom(F, G_1 \cup G_2)| = |\hom(F, G_1)| + |\hom(F, G_2)| \tag{3.5}$$

And products in a homomorphism can also be expanded:

$$|\hom(F, G_1 \times G_2)| = |\hom(F, G_1)| \cdot |\hom(F, G_2)| \tag{3.6}$$

Three types of homomorphisms and homomorphism densities can be related together with some effort.

$$|\hom_{\text{inj}}(F, G) = \sum_{F' \supseteq F} |\hom_{\text{ind}}(F', G)| \tag{3.7}$$

Using principle of inclusion and exclusion, we can also see that

$$|\hom_{\text{ind}}(F, G) = \sum_{F' \supseteq F} (-1)^{e(F')-e(F)} |\hom_{\text{inj}}(F', G)| \tag{3.8}$$

We also can relate general homomorphisms to injective homomorphisms. To accomplish this, we use partitions and the quotient graph.

Definition 3.6 (Quotient Graph)**.** The quotient graph F/P is formed combining all the vertices in each set in the partition into a singular vertex. Two vertices in F/P are adjacent if any of the original vertices were. All internal edges, if there were any, get replaced with a loop on the new vertex.

We now can express the relation between the two.

$$|\hom(F, G)| = \sum_{P} |\hom_{\text{inj}}(F/P, G)| \tag{3.9}$$

where P iterates through all graph partitions of the vertices of F.

Now we can turn to homomorphism densities rather than lengths of the sets of homomorphisms. These can be adapted from the previous equations.

The following is an easy derivation, as $\hom_{\text{inj}}$ and $\hom_{\text{ind}}$ are directly proportional to t_{inj} and t_{ind} and the normalizations do not differ:

$$\text{t}_{\text{inj}}(F, G) = \sum_{F' \supseteq F} \text{t}_{\text{ind}}(F', G) \tag{3.10}$$

$$\text{t}_{\text{ind}}(F, G) = \sum_{F' \supseteq F} (-1)^{e(F')-e(F)} |\,\text{t}_{\text{inj}}(F', G)| \tag{3.11}$$

Such that $V(F') = V(F)$

The next equation can be adapted given that a careful attention is paid to the different normalization being used for general homomorphism density and injective and induced homomorphism density. But we can use the fact that an injective homomorphism is the probably that a random injective map is a homomorphism and an a normal homomorphism is the probably any map is a homomorphism. The following mirrors [4].

Let f be a random map from $F \to G$ and let n be the number of vertices in F and k be the number of vertices in G.

If $t(F,G) \geq \mathrm{t_{inj}}(F,G)$, then:

$$\begin{aligned}
t(F,G) - \mathrm{t_{inj}}(F,G) &= \frac{|\hom(F,G)|}{n^k} - \frac{|\hom_{\mathrm{inj}}(F,G)}{n(n-1)\dots(n-k+1)} \\
&\leq \frac{|\hom(F,G)|}{n^k} - \frac{|\hom_{\mathrm{inj}}(F,G)}{n^k} \\
&= \text{Probability f is a homomorphism} - \text{Probability f in injective} \\
&= \text{Probability f is a non-injective homomorphism} \\
&\leq \text{Probability f is non-injective}
\end{aligned}$$

The probability two vertices are mapped to the same vertex in G is $\frac{1}{n}$, and $\binom{k}{2}$ is the number of ways to choose two vertices to test.

$$|t(F,G) - \mathrm{t_{inj}}(F,G)| \leq \frac{1}{n}\binom{k}{2} \tag{3.12}$$

If $t(F,G) \leq \mathrm{t_{inj}}(F,G)$, then:

$$\begin{aligned}
\mathrm{t_{inj}}(F,G) - t(F,G) &= \frac{|\hom_{\mathrm{inj}}(F,G)}{n(n-1)\dots(n-k+1)} - \frac{|\hom(F,G)|}{n^k} \\
&\leq \frac{|\hom_{\mathrm{inj}}(F,G)}{n^k} - \frac{|\hom(F,G)|}{n^k} \\
&= \hom(F,G)\left(\frac{1}{n(n-1)\dots(n-k+1)} - \frac{1}{n^k}\right) \\
&\leq n^k\left(\frac{1}{n(n-1)\dots(n-k+1)} - \frac{1}{n^k}\right)
\end{aligned}$$

$$|t(F,G) - \mathrm{t_{inj}}(F,G)| \leq \frac{n^k}{n(n-1)\dots(n-k+1)} - 1 \tag{3.13}$$

We can extend homomorphisms to weighted graphs

Definition 3.7 (Weighted homomorphism)**.** Let F be a simple graph and G be a weighted graph. Let the node weights be denoted as $\alpha_v(G)$ and the edge weights be denoted by $\beta_{uv}(G)$. Let each ϕ be a map such that $V(F) \to V(G)$. Let us define

$$\alpha_\phi = \prod_{u \in V(F)} \alpha_{\phi(u)}(G), \tag{3.14}$$

and

$$\hom_\phi(F,G) = \prod_{uv \in E(F)} \beta_{\phi(u)\phi(v)}(G). \tag{3.15}$$

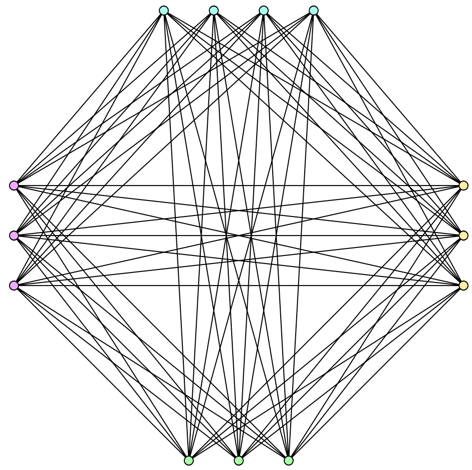

Figure 3. The Turán Graph, $T(13, 4)$

4 Bounding Theorems

There are many theorems that bound the homomorphism density of a graph, most of them include the complete graph.

The first one that will be explored is Turán's theorem, which is a partial solution to the Forbidden Subgraph problem. To state the theorem, an accompanying graph will have to be defined first.

Definition 4.1. The Turán graph $T(n, r)$ is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes that are as equal as possible and then connecting all vertices from different subsets.

Proposition 4.2. *This limits the number of edges to*

$$\left(1 - \frac{1}{r}\right)\frac{n^2 - s^2}{2} + \binom{s}{2}.$$

For simplicity's sake, this is often simplified to

$$\left\lfloor \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \right\rfloor.$$

This doesn't seem to be applicable to graph homomorphisms at first glance, but before the connection is drawn, a proof must be provided, as it is simpler to prove in this state than after it is rewritten in terms of homomorphism density.

Theorem 4.3 (Turán's Theorem)**.** *If K_{r+1} does not appear in a graph G of n vertices then it at most as many edges as the Turán graph $T(n, r)$. By proposition 4.2, this caps the maximal number of edges at*

$$\left(1 - \frac{1}{r}\right)\frac{n^2}{2}. \tag{4.1}$$

Proof. Additional proofs that may interest the reader can be found in [1]; for this paper, Turán's original proof is used.

Let us prove the theorem with induction on n. The inductive hypothesis would hold Turán's Theorem for a $n-1$ vertex graph.

Let G be a K_{r+1}-free graph with n vertices with the most edges possible. Seeing that G has as many vertices as possible, there must be a K_r subgraph of G. The r vertices from the subgraph into a set A, and put the other $n-r$ vertices into another set B. Now we can calculate a bound on the total number of edges.

There are $\binom{r}{2}$ edges in A because it is a complete subgraph. No vertex in B can connect to all of A, because then there would be a K_{r+1} subgraph. This caps the number of edges between A and B to $(|A|-1)(|B|) = (r-1)(n-r)$. The number of edges with B itself should be the number of edges in $T(n-r, r)$ at most due to the inductive hypothesis. This leads to a maximum of $\left(1-\frac{1}{r}\right)\frac{(n-r)^2}{2}$.

We find that the upper bound is

$$(r-1)(n-r) + \binom{r}{2} + \left(1-\frac{1}{r}\right)\frac{(n-r)^2}{2}.$$

This is indeed smaller than the limit from 4.1. ■

Now we can convert some statements from the original theorem to homomorphism related expressions.

To start, if there are no K_{r+1} subgraph in a graph G then $t(K_{r+1}, G) = 0$ because there is no non-adjacency for complete subgraphs. We can also see that $t(K_2, G)$ expands, by the definition of homomorphism density, to $\frac{|hom(K_2,G)|}{n^2}$ where n is the number of vertices in G. Due to the fact two mappings can map to the same edge with the vertices swapped we have to multiply the maximal number of edges from 4.3 by two.

Corollary 4.4. *If* $t(K_{r+1}, G) = 0$ *then* $t(K_2, G) \leq (1 - \frac{1}{r})$.

The Erdős–Stone theorem further extends Mantel's theorem but is less applicable to homomorphism density.

4.1 Triangle Density

Now a specific case of K_r would of course be K_3, although the term 3-clique will be used, also known as the triangle. There have been many attempts at finding limits as to the density of triangles in a graph. These are the most constrained limits on the relation between the density of triangles and the number of edges in a graph that are currently known.

Corollary 4.5 (Mantel's Theorem)**.** *If* $t(K_3, G) = 0$ *then* $t(K_2, G) \leq \frac{1}{2}$.

This is a corollary of theorem 4.3 that was discovered around 30 years prior. It provides a rough starting point for triangle density, even though its' predicate states that there are no triangles in the graph.

This was expanded by Goodman in 1959 to a proper bound of the lower limit of triangles given a number of edges.

Theorem 4.6 (Goodman's Theorem)**.**

$$t(K_3, G) \geq t(K_2, G)(2 \cdot t(K_2, G - 1) \tag{4.2}$$

Proof. See [3] for a complete proof. ■

The converse of 4.6 would be a special case of the Kruskal–Katona Theorem.

Theorem 4.7 (Kruskal–Katona Theorem)**.**

$$t(K_3, G) \leq t(K_2, G)^{\frac{3}{2}} \tag{4.3}$$

To prove this, we need to define a few more concepts to properly manipulate the adjacency matrices we will use.

Definition 4.8. Let A be a matrix $A\mathrm{v} = \mathrm{w} = \lambda\mathrm{v}$, and for each row, we define v to be the eigenvector of the linear transformation A and λ to be the eigenvalue of A.

Proof. Let us start with the graph G, which has n vertices. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of it's adjacency matrix, A_G. By the definition of a homomophism, $\hom(K_2, G) = t(K_2, G)(|V(G)|^2)$. We can convert the homomorphism density to a sum of eigenvalues of A_G,

$$t(K_2, G)(|V(G)|^2) = \sum_{i=1}^{n} \lambda_i^3 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)^{\frac{3}{2}} = \hom(K_2, G)^{\frac{3}{2}} \tag{4.4}$$

. ■

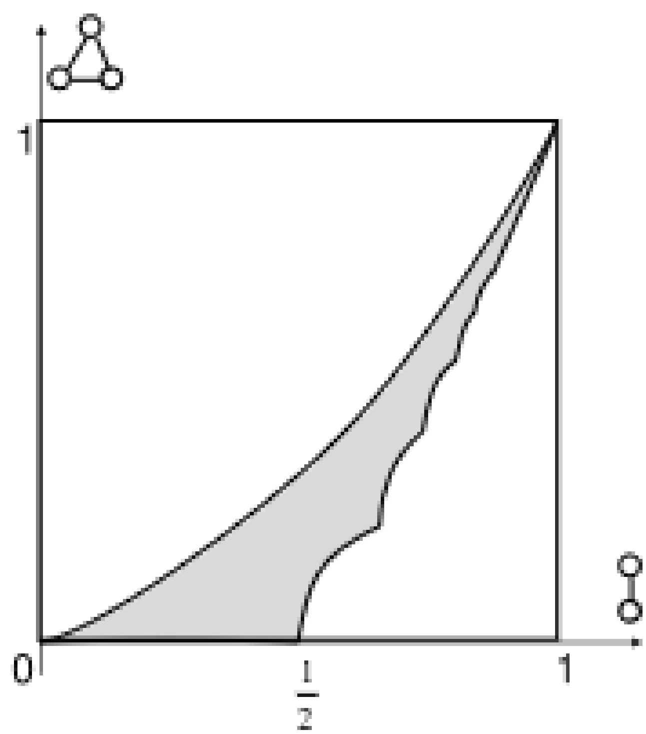

Figure 4. The density of edges vs. the maximal and minimal density of triangles

We cam chart the triangle density bounds, and get the following graph. The bottom is Goodman's theorem (4.6), while the top is the Krusal–Katona theorem (4.7). In this graph, the Krusal–Katona bound.

Prior to Goodman's theorem, the best lower bound was Bollobás theorem, which was a straight line from $\left(\frac{1}{2}, 0\right)$ to $(1, 1)$.

Theorem 4.9 (Bollobás Theorem)**.**

$$\sum_{i=1}^{n} a_i \cdot t(K_i, G) \geq 0 \tag{4.5}$$

is true for G if and only if it is true for K_m, where m is the number of vertices in G.

5 Useful connections

Homomorphisms and homomorphism density can be used to represent a wide variety of graph theory problems, a sampling of which have been added. Most of these examples have been adapted from [5] or [4].

Example (Colorings). We can see $\hom(G, K_i)$ is the number of colorings of the graph G with i colors such that no two adjacent nodes have the same color.

Definition 5.1 (Chromatic Number)**.** A chromatic number is the least number of colors needed to ensure that two adjacent vertices have different colors.

Example (Chromatic Number). If $\hom(K_2, G)$ is k-connected, then the chromatic number of G is at least $k + 3$.

Example (Walks). A walk in G is a homomorphism of a path into G, so $\hom(P_k, G)$ counts the number of walks with $k - 1$ steps in G.

Definition 5.2. An abelian group is one where the group operation is commutative.

Theorem 5.3 (S-Flows)**.** *If Γ is a finite abelian group and $S \in \gamma$ such that $S = -S$, and G is a graph. Then the S-flow is an mapping of an element in S to each edge with a specified orientation. Let $flo(G; \Gamma, S)$ denote the set of S-flows.*

$$flo(G; \gamma.S) = \hom(G, H) \tag{5.1}$$

where H is the complete looped directed graph on Γ.

Proof. A proof of this can be found in [2]. ■

Example (Moments of degree sequence). Homomorphisms from a star graph, S_k to G results in the sum of the moments of degree sequence.

$$|\hom(S_k, G)| = \sum_{i \in V(G)} \deg(i)^{k-1} \tag{5.2}$$

Example (Cycles and spectrum). Given any positive integer k, $\hom(C_k, G)$ is the trace of the k-th power of A_G, the adjacency matrix of the graph G.

$$\hom(C_k, G) = \mathrm{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k, \tag{5.3}$$

where $\lambda_1, \cdot, \lambda_n$ are the eigenvalues of A_G.

Example. We call a graph eulerian if all vertices are of an even degree. For example all cycle graphs are eulerian. Let this be a graph parameter that ranges from 0 to 1, where 1 would mean that the graph is eulerian. This is equivalent to $|\hom(., H)|$.

5.1 Cut Distance

Definition 5.4 (Normalized Maximum cut)**.** The Maximum cut, denoted as $\mathrm{Maxcut}(G)$, is an important graph parameter for dense graphs. It is the maximum number of edges between a set of the nodes of G and its complement. For our purposes, we will look at a normalized maximum cut. We define it to be as such:

$$\mathrm{maxcut}(G) = \frac{\mathrm{Maxcut}(G)}{|V|^2} = \max_{S \in V} \frac{e_G(S, VS)}{|V|^2} \tag{5.4}$$

Example. This does not seem to relate to homomorphisms at first glance, but what we can do is define a new graph H, edge-weighted on $\{1, 2\}$, with an edge-weight of 1 except on the non-loop edge, which would have a weight of 2. Then we see

$$2^{\mathrm{Maxcut}(G)} \leq \hom(G, H) \leq 2^{v(G)} 2^{\mathrm{Maxcut}(G)}.$$

We can easily simplify this by taking the log base 2, which yields us:

$$\mathrm{maxcut}(G) \leq \frac{\log_2(\hom(G, H))}{v(G)^2} \leq \mathrm{maxcut}(G) + \frac{1}{v(G)}. \tag{5.5}$$

Definition 5.5. We can extend the previous maxcut problem by involving partitions $q \geq 1$ instead of just 2. Instead of counting edges we can use a matrix B of coefficients $B_{ij}(i, j \in [g])$. We define the max multicut density as:

$$\mathrm{cut}(G, B) = \max \frac{1}{v(G)^2} \sum_{i,j} B_{ij} e_G(S_i, S_j) \tag{5.6}$$

Example. We can express this as homomorphism density:

$$\mathrm{cut}(G, B) \leq \frac{\log \hom(G, H)}{v(G)^2} \leq \mathrm{cut}(G, B) + \frac{\log q}{v(G)} \tag{5.7}$$

5.2 Graph Polynomials

Homomorphisms can be connected to graph polynomials in nuanced ways.

Definition 5.6. A graph polynomial is simply a graph parameter that is a polynomial

Definition 5.7. A multiset allows for repetition of elements.

Definition 5.8 (Multivariate Stable Set Polynomial)**.** Let G be a graph and $I(G)$ denote the set of independent subsets of the vertices of G. Let x_i represent each node i. For every multiset S of the nodes, let $x_s = \prod_{i \in S} x_i$. The multivariate stable set polynomial is

$$\text{stab}(G, x) = \sum_{S \in I(G)} x_S. \tag{5.8}$$

Example.

$$\text{stab}(G, 1, \ldots, 1) = \text{stab}(G) = \hom(G, H) \tag{5.9}$$

when H is the graph on two adjacent nodes, with a loop on 1.

We can also express $\hom(G, H)$ in terms of the intersection graph X of connected subgraphs G with 2 or more nodes.

Example. Given a simple graph G with a set V of vertices and E of edges, a weighted graph H, and a vector $t \in \mathbb{R}^{\text{Conn}(G)}$ such that $t_F = t(F, H)$, we have $t(G, H) = \text{stab}(L(\text{Conn}(G)), t)$.

6 Graphons

Actually computing the number of homomorphisms turns out to be tricky for large dense graphs. To help with this, we can introduce the concept of graphons.

Definition 6.1. A graphon, short for "graph function", also called a graph limit, is a symmetric measurable function W with domain $[0, 1] \times [0, 1]$ and range $[0, 1]$. To compute the graphon, let us find the graphon W_G for a graph G. Label the vertices of G $1, 2, 3, \ldots n$. For a simple graph, partition $[0, 1]$ into n equal intervals, each of length $\frac{1}{n}$. For each pair of vertices a and b, let $x \in [\frac{a-1}{n}, \frac{a}{n}]$ and $y \in [\frac{b-1}{n}, \frac{b}{n}]$. An example can be seen with figure 5.

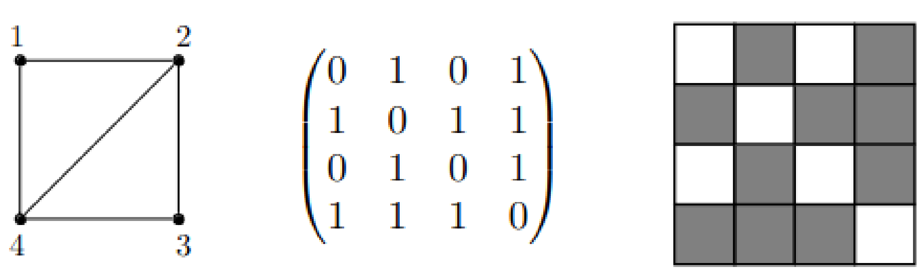

Figure 5. A graph, its adjacency matrix, and its graphon

Graphons are useful because they allow us to observe the limit of a dense graph as a function, which simplifies calculating homomorphism density, as we will see.

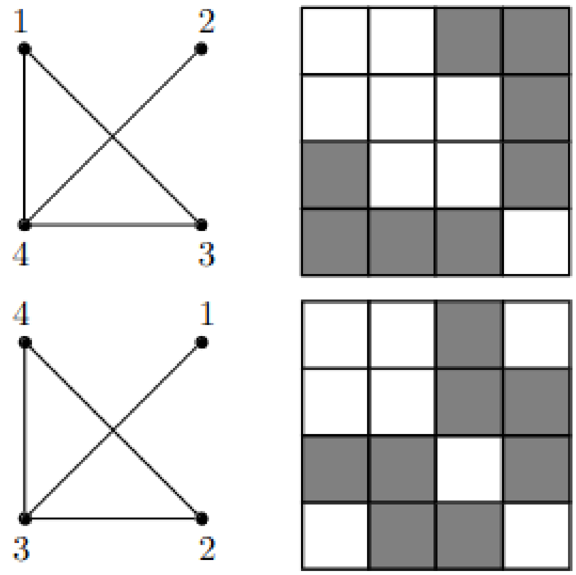

Figure 6. A graph labeled in two different ways resulting in two different graphons

As shown, by figure 6, two graphons can represent the same graph in very different ways by simply rearranging it's rows. Therefore we must define an equivalence between graphons.

Definition 6.2. We consider graphons W and Z to be weakly isomorphic if there exist preserving functions f and g such that $W(f(x), f(y)) = Z(g(x), g(y))$ almost everywhere. Note that the prior equation is often written as $W^f = Z^g$ for brevity.

Example. The two graphons in 6 are weakly isomorphic because we can write a function that rearranges one to get both in the same state. By this manner we can see that any relabeling of vertices results in weakly isomorphic graphons.

Definition 6.3. We can extend homomorphism density to a graphon W given a graph H like such:

$$t(H, W) = \int_{[0,1]}^{|V(H)|} \prod_{ij \in E(H)} W(x_i, x_j) \prod_{i \in V(H)} dx_i$$

.

Example. For example

$$t(K_3, W) = \int_{[0,1]^3} W(x, y)W(y, z)W(z, x)dxdydz$$

.

Injective homomorphisms are equivalent to regular homomorphisms when generalized to graphons because the probability $x_i = x_j$ is 0 since $x_i, x_j \in [0, 1] \in \mathbb{R}$.

Definition 6.4. Induced homomorphisms on graphons can be defined on graphons.

$$\mathrm{t_{ind}}(G, W) = \int_{[0,1]^{V(G)}} \prod_{ij \in E(G)} W(x_i, x_j) \prod_{ij \in \binom{V}{2} \setminus E(G)} (1 - W(x_i, x_j)) \prod_{i \in V(g)} dx_i \tag{6.1}$$

where $E(G)$ is the set of edges in G and $V(G)$ is the set of vertices in G. We define $\binom{V}{2} \setminus E(G)$ to be the set of all vertices that do not have an edge connecting them.

Proposition 6.5. *Two graphons, W and Z are weakly isomorphic if and only if $t(G, W) = t(G, Z)$ for every G.*

Proof. A proof can be found in [5]. ■

6.1 Generalizations

We can write corollaries to our relations:

Corollary 6.6.

$$\mathrm{t_{ind}}(F, W) = \sum_{F' \supseteq F} (-1)^{e(F') - e(F) \cdot t(F', W)} \tag{6.2}$$

where $V(F') = V(F)$

Corollary 6.7.

$$\mathrm{t_{inj}}(F, W) = \sum_{F' \supseteq F} \mathrm{t_{ind}}(F', W) \tag{6.3}$$

The limited theorems transfer over gracefully, but the Krusal–Katona theorem can be expressed in a slightly different form:

Corollary 6.8. *We can generalize theorem 4.7 to graphons and find that*

$$D_{2,3} = \{(t(K_2, W), t(K_3, W) : W \text{ is a graphon}\} \in [0, 1]^2 \tag{6.4}$$

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8 Bibliography

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