Irrationality and Transcendence: Advancing Beyond Algebra

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This Talk Will Explore:

1. The irrationality of e and a brief proof as an introduction
2. Some theorems used to establish transcendence
3. Schanuel's Conjecture, which can potentially be used to prove the transcendence of a lot of numbers
4. A hypothetical proof examining $e + \pi$'s transcendence if Schanuel's Conjecture was true

Background Information

1. Irrational: a number that cannot be represented as a ratio of integers
2. Transcendental: a subset of irrational numbers that do not satisfy any finite polynomial equation of rational coefficients. Transcendence implies irrationality.
3. For this talk, we will assume e and π are transcendental (except for the first proof), and use this notion to categorize a range of numbers.

e's Irrationality

This will be a simplified version of Fourier's proof of e's irrationality. First, we need an equation to manipulate, so let's define e in terms of it's Maclaurin Series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let's start by assuming the contradiction that $e = p/q$ when p and q are positive integers. When $x = 1$ we have:

$$e = \frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!}.$$

e's Irrationality (2)

Now that we have a new way of representing e, we can manipulate this equation until a contradiction is reached. We will start this process by first scaling both sides of the equation by a factor of $q!$.

$$\frac{q!p}{q} = q! + \frac{q!}{1!} + \frac{q!}{2!} + \frac{q!}{3!} + \dots \frac{q!}{n!}.$$

e's Irrationality (3)

The lefthand side of the equation must be an integer because it is a product of integer numbers. The righthand side, however, can be fundamentally split into two parts: a sum of integers, and a sum that equals some number S_n.

$$\underbrace{\frac{q!p}{q}}_{\in\mathbb{Z}} = \underbrace{q! + \frac{q!}{1!} + \frac{q!}{2!} + \ldots + \frac{q!}{q!}}_{\in\mathbb{Z}} + \underbrace{\frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \ldots}_{S_n}.$$

e's Irrationality (4)

1. S_n must be positive because p and q are positive
2. an infinite geometric series expansion G_n can represent an upper bound of S_n

We can define G_n to be the following and represent an upper bound.

$$S_n = \frac{1}{(q+1)} + \frac{1}{(q+2)(q+1)} + \ldots < G_n = \frac{1}{q+1} + \frac{1}{(q+1)^2} + \ldots$$

e's Irrationality (5)

G_n is just an infinite geometric series and can be summed in the following way.

$$G_n = \frac{\frac{1}{q+1}}{1 - \frac{1}{q+1}} = \frac{\frac{1}{q+1}}{\frac{q}{q+1}} = \frac{1}{q+1} \cdot \frac{q+1}{q} = \frac{1}{q}.$$

This gives us a upper and lower bound for S_n.

$$0 < S_n < 1$$

This gives us the contraction we need, and by the fundamental theorem of transcendental number theory, e cannot be rational so it must be irrational.

Powers of e

When considering powers more than $x = 1$, the r value causes the upper bound to diverge, so we need a more practical function. We do this using Legendre polynomials, function orthogonality and order of vanishing of polynomials.

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx$$

$$\int_{0}^{r} f(x)e^{x}dx$$

This proof will not be examined in this talk, but can be found in my paper.

Establishing Transcendence: Gelfond-Schneider Theorem

This theorem directly relates irrationality and transcendence and is used to categorize a large number of numbers in the form of a^b.

Theorem 0.1 (Gelfond-Schneider Theorem)

If both a and b are algebraic numbers, $a \notin \{0, 1\}$ and b is non-rational number, then any number in the form a^b is transcendental.

Theorems like this one helps us establish the transcendence of a very large scope of numbers, such as $2^{\sqrt{2}}$ (Gelfond–Schneider Constant), e^{π} (Gelfond's Constant) and i^i.

Schanuel's Conjecture

Conjecture 0.2

If we consider $z_1, z_2, \dots z_n$, which are complex numbers linearly independent on $\mathbb{Q}$, and their exponentials, $e^{z_1}, e^{z_2}, \dots e^{z_n}$ then, there are at least n algebraically independent numbers in:

$$z_1, z_2, \dots z_n, e^{z_1}, e^{z_2}, \dots e^{z_n}.$$

In other words, the field extension $\mathbb{Q}(z_1, z_2, \dots z_n, e^{z_1}, e^{z_2}, \dots e^{z_n})$, which has $2n$ terms, has transcendence degree at least n on $\mathbb{Q}$, provided that $z_1, z_2, \dots z_n$ has n terms that are complex and linearly independent.

Schanuel's Conjecture (2)

Definition 0.3 (Algebreic Independence)

If a and b are algebraically independent, there is no polynomial equation (with rational coefficients $\neq 0$) that will vanish when evaluated at these two numbers. In other words, algebraic independence is established when $P(a, b) \neq 0$, when $P(x, y)$ is the polynomial equation unless all the coefficients are zero.

e + π

Let $z_1 = 1$ and $z_2 = i\pi$, and consider the set of these complex numbers and their exponentials: $\{1, i\pi, e^{i\pi}, e^1\}$. Simplifying using Euler's Identity, we are left with $\{1, i\pi, -1, e\}$. By Schanuel's Conjecture, at least 2 of the 4 elements must be algebraically independent. Now let's assume the contradiction that $e + \pi$ is algebraic.

e + π (2)

This implies that it is a solution to a polynomial equation, when $a_0, a_1, \ldots a_n$ are rational coefficients.

$$a_n(e+\pi)^n + a_{n-1}(e+\pi)^{n-1} + \ldots a_1(e+\pi) + a_0 = 0$$

Now expanding the terms with $b_0, b_1, \ldots b_n$ being the coefficients that we get from expanding, we are left with:

$$b_n(\pi)^n + b_{n-1}(e)^n + \ldots b_2(\pi) + b_1(e) + b_0 = 0.$$

This equation shows a polynomial that links e and π algebraically, showing that they are not algebraically independent over $\mathbb{Q}$.

e + π (3)

Examining the transcendence degree:

1. -1 and 1 do not contribute to the degree because they can be linked through a polynomial equation ($x^2 - 1 = 0$, for example)
2. e and $i\pi$ do not contribute to the degree because the equation established on the last slide shows e and π are linked through a polynomial.

This conclusion violates Schanuel's Conjecture and shows that e and π must be algebraically independent, to make the transcendence degree 2 (which is necessitated by the conjecture), implying that $e + \pi$ is transcendental and in turn irrational.

Thank You

Thanks for listening, make sure to read my paper for more!

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