IRRATIONALITY AND TRANSCENDENCE: ADVANCING BEYOND ALGEBRA

ASHVATH RAJESH

Abstract. This paper examines two main properties: irrationality and transcendence. The paper first goes into a brief overview of some history as well as derivations and computations of our two main numbers: e and π. Then, it details a proof of e's irrationality as an introduction to irrationality proofs, and proves the irrationality of e^r and implements that notion towards π's irrationality. After establishing these proofs, the paper examines how transcendence can be established through a variety of means, and how establishing the irrationality and eventual transcendence of e and π can help prove the transcendence of other numbers, as well as how in other cases, approaching the establishment of transcendence directly can be used to imply irrationality. Some examples of generalized theorems explored in this paper include Baker's and Lindemann-Weierstrass Theorems. Then, the paper explores some problems the field is currently working on, such as $e + \pi$ and e^e, and how Schanuel's Conjecture could prove pivotal to proving such ideas. The paper also examines some applications, such as the study of transcendental functions as well as details the potential future of the field through an introduction to the notion of hypertranscendental functions and numbers.

1. Introduction

All of us are familiar with or have at least heard of the number π before. But what exactly is it? Some individuals may describe it as the ratio of a circle's diameter to its circumference, or describe it as a number that never ends, without repeating. Others describe it to be transcendental. But what exactly do any of these mean? Are there more numbers like this? This paper will focus on answering questions like these, as well as providing insight into why these statements are true through a variety of proofs, as well as exploring unsolved problems and applications of this field of mathematics.

When I was in middle school, calculating the area of circles, I was taught that π was equal to $\frac{22}{7}$. To the school's credit, it was sufficient at that time, but when my friend told me about how π is this absurd number that repeats infinitely without any patterns, I was fascinated by just imagining it. Pi was indeed what we call irrational, or a number that cannot be represented as a ratio of integers. Similar to π is e, another commonly known irrational number, that can be found everywhere in mathematics. Often described as Euler's number, the number derived from infinitely compounding, or the number that's unaffected by differentiation or integration, e, similar to π, is also transcendental. Everything from Euler's Identity to the Gaussian Integral includes these numbers somehow, and this phenomenon is no coincidence.

$$e^{i\pi} = -1,$$

Date: July 7th, 2024.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

So what is it that makes these numbers so special you may ask? The answer is transcendence: these numbers cannot be "solved" or "obtained" as a solution to a polynomial function of finite terms, i.e., they literally transcend algebra. Transcendental numbers must be irrational, but not all irrational numbers are transcendental ($\sqrt{2}$ is irrational but not transcendental). This is what sets these numbers apart from a square root of a non-perfect square, for example, because those numbers can be obtained as solutions to polynomials while numbers such as e and π simply cannot be obtained through algebraic means.

2. History

Our fascination with numbers began as early as 500 BCE, during the time of Pythagoras. During this time it was widely believed that all numbers could be represented as a ratio of integers: i.e. all numbers are rational. Humans stumbling upon irrational numbers, however, changed everything. Legend says that mathematician Hippasus was so shocked after discovering irrationality around $\sqrt{2}$ that he kept it a secret so it wouldn't contradict the mathematics of his time. Over the years, mathematicians learned more and more about irrational numbers, and Euclid even went on to prove that the number $\sqrt{2}$ was irrational.

Eventually, mathematicians was starting to encounter as well as ponder about numbers like π and e. Some famous names, like Newton and Leibniz, formulated ways to approximate π, through binomial theorems and infinite series, but there was still no solid proof regarding its irrationality. Johann Lambert was the first mathematician to prove π's irrationality, which he did using continued fractions. Later, mathematician Joseph Fourier developed the famous Fourier series, which is extremely prominent in mathematics, and established the notion that irrational numbers are prevalent in trigonometry and calculus, setting into perspective how important the study of these numbers really is.

The 19th and 20th centuries were when we started exploring the idea of transcendent numbers, which is a subset of irrational numbers. Joseph Liouville was the first to explore this idea and lay out proofs regarding this concept. The first transcendence proof, however, was regarding e and was done by mathematician Charles Hermite, which then also implied that e was irrational. Pi was quick to follow, and its transcendence was proven by Ferdinand von Lindemann in 1882. Over the years, people have experimented extensively with these numbers and developed different and more concise proofs, as well as exploring the transcendence of a combination of these numbers (like $e + \pi$ or e^π). This is still ongoing today and is an active field of mathematical study.

This paper will start with the proofs of e's irrationality, as a sort of introduction to irrationality proofs, and then explore e^r as well as π's irrationality. Then, we will go into transcendence, as well as more implications regarding numbers of this sort. It is important to note that this paper's irrationality proofs flow similarly to those of Timothy Y. Chow's paper [4], which is inspired by mathematician Ivan Niven's [7] proof of π's irrationality.

3. e AND π AS NUMBERS

The first thing one might wonder when encountering numbers such as e and π, is how they are computed or derived. This section will work to briefly address those questions by exemplifying some standard ways of approximating these numbers, which is pivotal in the scope of irrationality and transcendence.

$$e \approx 2.7182818284\ldots$$

$$\pi \approx 3.1415926535\ldots$$

Starting with e, perhaps the most common way of representing e is through the following limit:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n.$$

In addition to this, however, mathematicians have also used things like Maclaurin series expansions (gone over later in this paper), integrals, derivatives, as well as probability functions to define e.

$$\int e^x = e^x + C$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\int_1^e \frac{dx}{x} = 1.$$

Pi however, can be more simple or complex (depending on how you interpret it). As everybody knows, π can be defined through a circle's area or circumference, as exemplified below.

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.$$

This, however, requires one to inscribe an infinite-sided polygon in the circle and trying to compute its perimeter to obtain an expression to approximate π. This process is quite tedious and over the years mathematicians have found other ways to represent this number. Nevertheless, there are a multidude more alternative ways to approximate π, such as Newton's use of Pascal's triangle. Similarly, another one of many solutions to this problem was outlined by Srinivasa Ramanujan [3], who proposed the following summation which converges rapidly, helping approximate more digits of π.

$$\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4(396)^{4k}} = \frac{1}{\pi}.$$

These are just some of the many ways mathematicians represent these numbers, and it's no exaggeration that they appear everywhere in math since it's evident by the fact that there are so many ways these numbers can be represented and computed. Now that we have been introduced to where these numbers come from, we can now prove their properties and look into their potential implications.

4. The Irrationality of e

Firstly, we will be dealing with the number e, and discussing its irrationality. The way we do this is by setting up a proof by contradiction and assuming the exact opposite of what we desire, in other words, assuming that e is a rational number. Rational numbers, by definition, can be represented as the quotient of two integers, which gives us $e = p/q$ when p and q are positive integers. This method of proving e's irrationality goes by the name of Fourier's proof [4].

Now, we need to find a contradiction or some kind of manipulation of this relationship that creates problems, which would lead to the conclusion that e cannot possibly be rational, and must therefore be irrational. To get there, we will start by representing e and its Maclaurin series expansion, which approximates it using a polynomial of infinite terms with coefficients that correspond to the derivatives of e^x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Using the Maclaurin expansion of e^x, where $x = 1$, we can express our number e differently.

$$e = \frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!}. \tag{4.1}$$

Now that we have a new way of representing e, we can manipulate this equation until a contradiction is reached. We will start this process by first scaling both sides of the equation by a factor of $q!$.

$$\frac{q!p}{q} = q! + \frac{q!}{1!} + \frac{q!}{2!} + \frac{q!}{3!} + \dots \frac{q!}{n!}.$$

The lefthand side of the equation must be an integer because it is a product of integer numbers. The righthand side, however, can be fundamentally split into two parts: a sum of integers, and a sum that equals some number S_n.

$$\underbrace{\frac{q!p}{q}}_{\in\mathbb{Z}} = \underbrace{q! + \frac{q!}{1!} + \frac{q!}{2!} + \dots + \frac{q!}{q!}}_{\in\mathbb{Z}} + \underbrace{\frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \dots}_{S_n} \tag{4.2}$$

$$\frac{q!p}{q} = q! + \frac{q!}{1!} + \frac{q!}{2!} + \dots + 1 + \frac{1}{(q+1)} + \frac{1}{(q+2)(q+1)} + \dots$$

Upon observing the terms on the right-hand side, one may observe that until the q'th term, the expansion represents an integer: the only operations done between these numbers are products and finite sums of integers, so the result must also be an integer, similar to the left-hand side of the equation. The remaining terms, the sum of which is labeled S_n, can be represented through an upper and lower bound. These bounds are created through two main ideas:

(1) S_n must be positive because p and q are positive

(2) an infinite geometric series expansion G_n can represent an upper bound of S_n

The lower bound for this is quite straightforward: since S_n is always positive, it is greater than zero. For the upper bound, however, we will compare it to the geometrically modeled equation G_n, and have it serve as an upper bound because it is always larger than S_n.

$$S_n = \frac{1}{(q+1)} + \frac{1}{(q+2)(q+1)} + \ldots < G_n = \frac{1}{q+1} + \frac{1}{(q+1)^2} + \ldots \tag{4.3}$$

One advantage of comparing to a geometric model is the fact that the sum of an infinite number of terms can be computed easily, so long as it's common factor $|r| < 1$, which our equation satisfies: $r = \frac{1}{q+1} < 1$ because q is a positive integer. Given this fact, the infinite sum of G_n can be computed using the formula:

$$G_n = \frac{a_1}{1-r}.$$

Knowing that $a_1 = \frac{1}{q+1}$ and $r = \frac{1}{q+1}$, plunging into the equation above yields the following upper bound:

$$G_n = \frac{\frac{1}{q+1}}{1 - \frac{1}{q+1}} = \frac{\frac{1}{q+1}}{\frac{q}{q+1}} = \frac{1}{q+1} \cdot \frac{q+1}{q} = \frac{1}{q}.$$

Recall that q is a positive integer, so the maximum value that $1/q$ can take on is 1, which will serve as an upper bound. This results in the inequality:

$$0 < S_n < 1. \tag{4.4}$$

Now, an issue has emerged: since our expression only involves the sum and product of integers, S_n must also be an integer. This, however, is not true because there are no integers between 0 and 1. Although this seems like common sense, the more formal definition of this phenomenon also exists and is described by the fundamental theorem of transcendental number theory. This contradicts our proof that e is a rational number, meaning it must be irrational. This proof is a rather simple example, and serves as a great example to outline the main strategy mathematicians use in irrationality proofs. In the next section, we will expand on the same fundamental ideas, and apply them to a much more complicated problem: the irrationality of e^r, and eventually to π.

5. Expanding to Powers of e

Before exploring the irrationality of π, it is important to consider the powers of e. Not only is the proof for e^r extremely identical to that of π (which aids us in the future), but it also shows us a significant mathematical fact about the irrationality of the powers of e that's vastly applicable in mathematics. In our original expression, consisting of e^x, what if $x \neq 1$? In this case, our upper bound for S_n using a geometric series G_n fails because the expansion of S_n is not converging for $|r| > 1$. The Maclaurin series expansion is no longer sufficient for this proof, and we need a new expression involving e^r which provides a bound for us to make a contradiction even when $r \neq 1$. This is where Legendre polynomials come in.

For the sake of this proof, we will not be exploring the deeper mathematical connections to Legendre polynomials, but it is still important to know that Legendre polynomials are a system of orthogonal polynomials, as illustrated below. Additionally, they greatly simplify the coefficients of calculation, which is extremely useful in our proof, which uses terms from the series. In other words, since we need to manipulate functions with respect to coefficients, Legendre polynomials help keep our job straightforward.

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx.$$

It is useful to note that we have picked Legendre Polynomials out of all the other orthogonal polynomials because of simplicity: $w(x)$, or the weight function, is just $= 1$ for Legendre Polynomials, simplifying the result to just two functions inside the integral, which is really all we need for this proof.

$$\int_{a}^{b} f(x)g(x)w(x)dx = 0.$$

We will utilize this integral, with $g(x) = e^x$ and $w(x) = 1$ and to construct an optimal equation for $f(x)$ to help obtain our contradiction. For now, we will let $f(x)$ be some polynomial with integer coefficients and some degree k. Additionally, since our area of interest is $[0, r]$, we can alter the bounds of the integral to fit that interval.

$$\int_{0}^{r} f(x)e^x dx. \tag{5.1}$$

Now, we need an equation, so let's start by integrating the above integral. We will assume $f(x)$ is just some polynomial (which we will choose later) and integrate by parts. Also, since k is a finite number, we don't need to worry about a divergence. After integrating one time we are left with the following:

$$\int_{0}^{r} f(x)e^x dx = [f(x)e^x]_0^r - \int_{0}^{r} f'(x)e^x dx. \tag{5.2}$$

Continuing, the second integration leaves us with:

$$\int_{0}^{r} f(x)e^x dx = [f(x)e^x]_0^r - [f'(x)e^x]_0^r + \int_{0}^{r} f''(x)e^x dx,$$

$$\int_{0}^{r} f(x)e^x dx = [e^x(f(x) - f'(x))]_0^r + \int_{0}^{r} f''(x)e^x dx.$$

We can now determine a pattern and represent this integral's solution as a sum, recalling that it isn't infinite because our polynomial $f(x)$ has a fixed degree. Still, we'd have to keep going to reach a solution. Recall that if we take the n'th order (or higher) derivative of a polynomial of degree $(n-1)$, then its derivative is zero. As we keep integrating, the orders of the derivatives in the integral eventually equals zero, so the integral on the right-hand side of the equation is just zero. Now we just need to evaluate the remaining terms over $[0, r]$. If we define:

$$F(x) = f(x) - f'(x) + f''(x) - \dots$$

which alternates signs because of integration by parts, then the integral evaluates to:

$$\int_{0}^{r} f(x)e^x dx = F(r)e^r - F(0). \tag{5.3}$$

Now, we have an equation revolving e^r, but we still need to manipulate it to get a contradiction. For this proof, we will assume $e^r = p/q$. By substituting this and multiplying by q on both sides and dividing by some $n!$ for a sufficiently large n (the reason for this is examined closely in the steps to follow), we obtain:

$$\frac{q}{n!}\int_0^r f(x)e^x dx = \frac{F(r)p}{n!} - \frac{F(0)q}{n!}. \tag{5.4}$$

The next step is to use a similar contradiction as the proof for when $r = 1$ in e^r: we need to get the same setup to utilize the fundamental theorem of transcendental number theory. To achieve this goal, we will need to convince ourselves that the two terms on the righthand side of the equation, $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$, are integers. The way we do this involves the definition we set for $F(x)$. Since it involves taking the derivative repeatedly, the coefficient term must be factorial-like, assuming $f(x)$ started with integer coefficients, which we can assure when picking the equation. The exact equation for $f(x)$ to get a solution is something we will optimize later, but for now, we can assume that $f(x)$ has integer coefficients to go forward with our proof.

Lemma 5.1. *The coefficients of $f^{(n)}(x)/n!$ are integers ($n > 0$) if $f(x)$ has integer coefficients.*

Proof. Consider $f(x) = x^a$. Then, $f^n(x) = (a)(a-1)\ldots((a-n)+1)x^{a-n}$. If we were to simplify this, we obtain:

$$f^{(n)}(x) = \frac{a!}{(a-n)!}x^{a-m}.$$

Now dividing by $n!$, we have

$$\frac{f^{(n)}(x)}{n!} = \frac{a!}{(a-n)!n!}x^{a-m},$$

leaving our coefficient to be $\binom{a}{n}$. Both a and n are integers (these are values picked by us), so our coefficient $\binom{a}{n}$ is therefore also an integer. ■

We can now use this idea and apply it to our two areas of interest, $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$. Before we do this, however, we need to consider the degree of our polynomial $f(x)$. Let's define the degree of this polynomial to be some integer k. From our definition: $F(x) = f(x) - f'(x) + f''(x) - \ldots + f^{(k)}(x)$ (the sign before $f^k(x)$ can be positive or negative depending on the degree, but the main idea here is that it alternates signs). Now our proof applies if $k \geq n$ since our equation becomes:

$$\frac{f^{(k)}(x)}{n!} = \binom{k}{n}x^{k-n}.$$

If $k < n$ however, we have a problem. To help against these lower degree k values, we will manipulate $f(x)$ so that Lemma 5.1 still applies.

Now to do this, we will set polynomial $f(x)$ as some function that vanishes at n'th order at our points of interest, which in this case are our bounds of integration, $x = 0$ and $x = r$. One such option is the polynomial $f(x) = x^n(r-x)^n$. It is important to understand that we do this because $f(x)$ is a polynomial that vanishes to order n at both of these points.

Definition 5.2 (Order of Vanishing)**.** A polynomial $f(x)$ is said to vanish at $x = a$ if $f(a) = 0$ when $a \in \mathbb{R}$, and its order n_1 (The n from the definition is not to be confused with the n from our equation so we will denote it as n_1) is defined as the smallest possible integer so that $f^{n_1}(a) \neq 0$. The degree to which it vanishes at $x = a$ is determined using the following limit when n_1 is the order of vanishing so long as the limit takes on a non-zero finite value.

$$\lim_{x\to a} \frac{f(x)}{x^{n_1}} \neq 0,$$

$$\lim_{x\to a} \frac{f(x)}{x^{n_1}} \neq \pm\infty.$$

In our case, our equation is $x^n(r-x)^n$. If we were to examine our equation near our bounds $x = 0$ and $x = r$, we can see the following:

$$\lim_{x\to 0} \frac{x^n(r-x)^n}{x^{n_1}}.$$

For $x = 0$, it is only possible for the limit to be a non-zero finite if $n_1 = n$, meaning that our order of vanishing at $x = 0$ is n.

$$\lim_{x\to r} \frac{x^n(r-x)^n}{x^{n_1}}.$$

Similarly, at $x = r$, the only possible way to keep the equation as a non-zero finite is to have $n_1 = n$ which means $f(x)$ vanishes at order n to both of our bounds of integration.

From this, we know $n_1 = n$ since the smallest order derivative where $f(x) = x^n(r-x)^n$ does not equal zero is n. Additionally, this implies that for any number k, when $k < n$, $f^k(x) = 0$ because n, by definition, is the smallest possible integer so that $f^{(n)}(x) \neq 0$, and anything smaller has a 0 derivative. We now can use this fact to prove that both $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$ are integers.

Corollary 5.3. *If $f(x) = x^n(r-x)^n$, when r and n are integers ($n \geq 0$), and $F(x) = f(x) - f'(x) + f''(x) - \ldots$, then $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$ are integers.*

Proof. There are 2 cases to this, $k \geq n$ and $k < n$

(1) if $k \geq n$, then by Lemma 5.1, $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$ are both integers.
(2) if $k < n$ then $f^{(k)}(0) = f^{(k)}(r) = 0$ using Definition 5.2, so $F(x)$ would just be 0, which is an integer, showing that $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$ are both integers.

This shows that $\frac{F(r)p}{n!}$ and $\frac{F(0)q}{n!}$ are both either 0 or some combination of integers, both of which tell us that our 2 terms are integers. ∎

Now taking a look back at our original integral:

$$\underbrace{\frac{q}{n!}\int_0^r f(x)e^x dx}_{\in[0,1]} = \underbrace{\frac{F(r)p}{n!} - \frac{F(0)q}{n!}}_{\in\mathbb{Z}}.$$

We know that the right-hand side of the equation is a difference of integers, but we also know that the left-hand side of the equation is non-zero because $r > 0$. We also know that the left-hand side of the equation will be less than one for any sufficiently large n because $n!$ grows much faster than any other function. A combination of these facts tells us that the left-hand side is between 0 and 1, and the right-hand side is an integer: we have reached the same contradiction as we did when $x = 1$ in e^x, and using the fundamental theorem of transcendental number theory, we can contradict this method and prove that e^r for any power $r > 0$ is irrational. But wait! We can also say that e^r is irrational when $r < 0$ because any negative powers just result in $\frac{1}{e^r}$ which still cannot be represented as a ratio of integers, making e^r irrational for any $r \neq 0$.

6. The Irrationality of π

Now that we have made it to the irrationality of π, we will need to tweak our methods slightly. To fit our purpose, we will first change e^x to $\sin x$ due to its close relationship with π. But $\sin x$ itself is what we call a transcendental function: its very definition is based on numbers like π because there is no algebraic way to represent it. Nevertheless, $\sin x$ can be defined through a Maclaurin Series expansion.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Using this definition we get an approximation for $\sin x$ as a polynomial, and we can use this to define π to be the smallest positive solution to $\sin x = 0$. In this way, $\sin x$ is closely correlated with π and is the reason we will be substituting it in place of e^x. Additionally, we will be replacing the bounds of integration from $[0, r]$ to $[0, \pi]$ and assuming the contradiction that $\pi = a/b$ when a and b are integers. Doing all of this to our integral results in the following:

$$\int_0^{\pi} f(x)\sin(x)dx. \tag{6.1}$$

Now, we will need to again define $F(x)$ which we get as a result of integration by parts. Also, we can substitute in $\pi = a/b$.

$$\int_0^{a/b} f(x)\sin(x)dx = [-f(x)\cos(x)]_0^{a/b} + \int_0^{a/b} f'(x)\cos(x)dx.$$

With repeated integration by parts, we are left with:

$$\int_0^{a/b} f(x)\sin(x)dx = [-f(x)\cos(x) + f'(x)\sin(x) + f''(x)\cos(x)]_0^{a/b} - \int_0^{a/b} f^{(3)}(x)\cos(x)dx.$$

You may notice that in this case, we cannot factor, like we did with e^x. We can, however, simplify to a large extent due to the bounds. Every term has either a $\sin x$ or a $\cos x$ attached to it, and in $x = 0$ and $x = \pi$, these functions are either 0, or ± 1. In our case, all the even-order derivatives stay, since $\sin(0)$ and $\sin(\pi)$ both evaluate to zero. Using this fact, we can define a new $F(x)$.

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \dots$$

Using this definition, our integral can be represented in terms of $F(x)$.

$$\int_0^{a/b} f(x)\sin(x)dx = F(\pi) + F(0). \tag{6.2}$$

Next, we will scale up this integral by a factor of $\frac{b^n}{n!}$ on both sides and then prove $\frac{b^nF(a/b)}{n!}$ and $\frac{b^nF(0)}{n!}$ are integers.

$$\frac{b^n}{n!}\int_0^{a/b} f(x)\sin(x)dx = \frac{b^nF(a/b)}{n!} + \frac{b^nF(0)}{n!}. \tag{6.3}$$

We now need to modify our $f(x)$, which we will change to be equal to $f(x) = x^n(a - bx)^n$. The reason for this is seen shortly, but using this, we can get to our contradiction. This equation vanishes to order n at $x = 0$ and $x = a/b$, and we can get this solution using our Definition 4.2's limit.

Corollary 6.1. *If $f(x) = x^n(a - bx)^n$, when r, a, and b are integers (for $b \neq 0$ and $n \geq 0$), and $F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \dots$, then $\frac{b^nF(a/b)}{n!}$ and $\frac{b^nF(0)}{n!}$ are integers.*

Proof. Again, there are two cases to this, $k \geq n$ and $k < n$.

(1) if $k \geq n$, then by Lemma 5.1, $\frac{b^nF(a/b)}{n!}$ and $\frac{b^nF(0)}{n!}$ are integers (remember b^n is an integer so scaling by it does not affect the results)

(2) if $k < n$ then $f^{(k)}(0) = f^{(k)}(a/b) = 0$ using Definition 5.2, so $F(x)$ would just be 0, which is an integer, showing that $\frac{b^nF(a/b)}{n!}$ and $\frac{b^nF(0)}{n!}$ are both integers. Here b^n scaling does not change anything either.

■

It is important to note that the scaling by b^n is not required to prove these, since differentiation tells us that the term $\frac{F(a/b)}{n!}$ is an integer, but we have the scaling there if one decides to prove using the reasoning that evaluating a polynomial with integer coefficients at $x = a/b$ and then scaling results in an integer. This is a different way to do it but has the same result, and the proof has this included to be flexible to some extent. That aside, we have now gotten the same contradiction that we had used the Fundamental Theorem of transcendental number theory on for e^r.

$$\underbrace{\frac{b^n}{n!}\int_0^{a/b} f(x)\sin(x)dx}_{\in[0,1]} = \underbrace{\frac{b^nF(a/b)}{n!} + \frac{b^nF(0)}{n!}}_{\in\mathbb{Z}}. \tag{6.4}$$

Similar to e^r, since the equation on the right-hand side is an integer and the left-hand side is between 0 ($a/b > 0$) and 1 ($n!$ grows fastest), we can say the equation is contradicted by the fundamental theorem of transcendental number theory and therefore conclude that π is also irrational.

7. Establishing Transcendence

So far in the paper we have examined the irrationality of e, e^r, and π and the proofs that result in their irrationality. Irrationality is extremely important to establish, especially in the scope of transcendence, because all transcendental numbers by definition are irrational. In addition to not being able to be represented as a ratio, these numbers also cannot be obtained as a solution to an algebraic polynomial of finite terms: they are transcendental. Irrationality, as shown later in this paper, is a property that is extremely important for establishing transcendence. Additionally, if the irrationality of a function cannot be proven, approaching the transcendence proof could benefit us due to all transcendental numbers being irrational. The proof of transcendence is similar to that of irrationality and involves the assumption that e or π can be obtained as a solution to a polynomial to get a contradiction. However, these proofs are extremely tedious and not the focus of this paper, and it will explore more applicable forms of proving transcendence through generalized theorems that help us establish the transcendence of a large set of numbers at once.

Before going into the theorems of this section, it is important to note that there are a few mathematical facts we use here that were not presented in this paper. This includes the fact that π^r when $r \in \mathbb{Z}$ provided that $r \neq 0$ is irrational, and the fact that e and π are transcendental. Although these proofs are not showcased, they can be done using a series of polynomial equations, setting these numbers as the solution, and then manipulating them until a contradiction is reached, or using a more technical approach using repeated products. When multiple of these numbers are combined though, through multiplication addition, exponentials, etc., we need some more generalized guidelines to categorize these types of numbers (going through these intensive proofs is simply not practical), which is the primary focus of this section.

Remark 7.1. Although the proof of π and e's transcendence is not showcased specifically, they can be confirmed to be transcendental through the theorems to follow.

7.1. Gelfond–Schneider Theorem.
This theorem, developed by Aleksandr Gelfond and Theodor Schneider, directly relates irrationality and transcendence and is used to categorize a large number of numbers in the form of a^b [10].

Theorem 7.2. *If both a and b are algebraic numbers, $a \notin \{0, 1\}$ and b is non-rational number, then any number in the form a^b is transcendental.*

This helps us establish the transcendence of a very large scope of numbers, and we will be examining some of the examples below. It is very important to note that most of the theorems in this section rely on the idea of irrationality which we already proved: this is a basis to apply many of these theorems and knowing which numbers are irrational vs. those that are transcendental is a necessity.

(1) $2^{\sqrt{2}}$ (Gelfond–Schneider Constant)

Proof. $a = 2$ is algebraic and $\neq 0$ or 1, $b = \sqrt{2}$ is irrational, so $2^{\sqrt{2}}$ is transcendental. ∎

(2) e^{π} (Gelfond's Constant)

Proof. Rewriting, we have $e^{\pi} = (e^{i\pi})^{-i}$ and $e^{i\pi} = -1$ by Euler's identity, so a in this case is a not zero or one and $b = i$ is non-rational (it doesn't have to be irrational, just not rational, which i satisfies because imaginary numbers are never rational), so e^{π} is transcendental. ■

(3) i^i

Proof. $i = \sqrt{e^{i\pi}} = e^{\frac{i\pi}{2}}$, so $i^i = (e^{\frac{i\pi}{2}})^i$. This is just equal to $e^{-\pi/2}$, which we know is irrational through the previous proof of e^{π} (we just need to use Euler's Identity again). ■

7.2. **Lindemann–Weierstrass Theorem.** This theorem, also called the Hermite Lindemann Weierstrass theorem, was proposed by Ferdinand von Lindemann, Karl Weierstrass, and Charles Hermite. This theorem uses number theory to classify a wide range of numbers, as is based on linear independence [8]. Let's consider the numbers a, b and c. Now consider three different algebraic coefficients k_1, k_2, and k_3. Their linear combination is:

$$k_1 a + k_2 b + k_3 c = 0$$

If there are no possible k_1, k_2 and k_3 other than 0, then the numbers a, b and c are said to be linearly independent over $\mathbb{A}$ (the algebraic numbers).

Theorem 7.3. *If the numbers $a_1, a_2, a_3, \ldots$ are distinct algebraic numbers, then the numbers $e^{a_1}, e^{a_2}, e^{a_2}, \ldots$ are linearly independent over $\mathbb{A}$.*

This theorem increases the scope of transcendental numbers that can be proven drastically, and also helps us with outputs of a wide number of transcendental functions, like the trigonometric ($\sin x, \cos x, \tan x$) and hyperbolic trigonometric functions ($\sinh x, \cosh x, \tanh x$), and the natural logarithmic function ($\ln x = \log_e x$). More complex transcendental functions will be talked about later in the paper. That being said, this theorem has helped us establish transcendence for numbers such as those categorized below.

(1) e^a when $a \in \mathbb{A}$ and $a \neq 0$.

Example. $e^{\sqrt{2}}, e^{3\sqrt[3]{2}}$, etc.

(2) $e^{k\pi}$ when $k \in \mathbb{A}$ and $a \neq 0$

Example. $e^{3\pi}, e^{\pi\sqrt{d}}$ for positive integer d, etc.

(3) Evaluation of transcendental functions

Example. $\sin(1), \cosh(67), \ln(14)$, etc.

(4) Exponential Logarithms $e^{ln(d)}$

Example. $e^{ln(d)}$ for any positive integer d, etc.

(5) Imaginary exponentials

Example. $e^{i\sqrt{2}}$, etc.

7.3. **Baker's theorem.** This theorem, developed by Alan Baker, gives us a computable bound for linear combinations involving logarithms [2]. The part of his theorem we will focus on is a simplification, but still sufficient for the purpose of this paper.

Theorem 7.4. *If $\alpha_1, \alpha_2, \ldots \alpha_n$ are non-zero algebraic numbers, and $\lambda_1, \lambda_2, \ldots \lambda_n$ are algebraic (all of them cannot equal zero), then $\lambda_1 \ln(\alpha_1) + \lambda_2 \ln(\alpha_2) + \ldots \lambda_2 \ln(\alpha_2)$ is 0 or transcendental, but if any λ is not rational, the form is transcendental.*

Example. Let $\alpha_1 = 2$, $\alpha_2 = 3$ and $\lambda_1 = 1$ and $\lambda_2 = -1$, then we have:

$$\ln(2) - \ln(3)$$

Since we know this equation $\neq 0$, then $\ln(2) - \ln(3)$ must be transcendental by Baker's Theorem.

This theorem explores combinations of logarithms, and has a wide variety of applications in addition to transcendence as well, and usually deals with the transcendence of combinations of logarithmic function outputs. It also connects with the notion of proving irrationality first to a considerable extent, because the fact that a number was irrational can be used alongside Baker's Theorem to confirm it's transcendence. Some of these applications are detailed below.

(1) Transcendence of combinations of logarithms and equations involving logarithms
(2) Diophantine Equations which involve variables in the exponent (such as $a^x - b^y = c$), which logarithms can help simplify.
(3) Linear Combinations involving logarithms and their potential results and implications

8. Unsolved Problems and Current Areas of Study

Using either, or a combination of any of the theorems presented in this paper gives us a great way to classify a large range of numbers and categories of these numbers in combination. There is, however, a great deal we do not know about transcendental and irrational numbers, including proofs of certain combinations of irrational numbers. Amongst these are $e + \pi, e^e, e\pi$, and many more. Nevertheless, one interesting yet potentially significant mathematical conjecture to consider is Schanuel's.

This conjecture is extremely important to consider in the subject of transcendence because of how it guarantees the transcendence of certain numbers, such as $e + \pi$ which we do not know at the moment. In fact, we do not even know if $e+\pi$ is irrational, and its transcendence could imply its irrationality.

What we do know at this time, is that at least one, if not both of $e + \pi$ and $e\pi$ must be irrational. If we construct a polynomial with irrational roots e and π, then it this implies that at least one of the coefficients must be irrational. As illustrated below, we can see that the coefficients are 1, $e + \pi$ and $e\pi$. Since we know 1 is rational, at least one of $e + \pi$ and $e\pi$ must be irrational.

$$(x - e)(x - \pi) = 0,$$

$$(1)x^2 - (e+\pi)x + e\pi.$$

If Schanuel's Conjecture were verified to be true, we would be able to prove $e+\pi$'s transcendence and confirm its irrationality.

8.1. **Schanuel's Conjecture.** This conjecture proposed by Stephen Schanuel in the 1960s has not been able to be proven or disproven to this day but uses elements of Gelfond–Schneider and Lindemann–theorems to make a hypothesis that, if proven, could help prove the transcendence of a myriad of numbers [9].

Conjecture 8.1. *If we consider $z_1, z_2, \ldots z_n$, which are complex numbers linearly independent on $\mathbb{Q}$, and their exponentials, $e^{z_1}, e^{z_2}, \ldots e^{z_n}$ then, there are at least n algebraically independent numbers in:*

$$z_1, z_2, \ldots z_n, e^{z_1}, e^{z_2}, \ldots e^{z_n}.$$

In other words, the field extension $\mathbb{Q}(z_1, z_2, \ldots z_n, e^{z_1}, e^{z_2}, \ldots e^{z_n})$, which has $2n$ terms, has transcendence degree at least n on $\mathbb{Q}$, provided that $z_1, z_2, \ldots z_n$ has n terms that are complex and linearly independent.

Definition 8.2 (Algebreic Independence)**.** If a and b are algebraically independent, there is no polynomial equation (with rational coefficients $\neq 0$) that will vanish when evaluated at these two numbers. In other words, algebraic independence is established when $P(a,b) \neq 0$, when $P(x,y)$ is the polynomial equation unless all the coefficients are zero.

Conjecture 8.3. *$e+\pi$ is irrational assuming Schanuel's Conjecture is true.*

Proof. Let $z_1 = 1$ and $z_2 = i\pi$, and consider the set of these complex numbers and their exponentials: $\{1, i\pi, e^{i\pi}, e^1\}$. Simplifying using Euler's Identity, we are left with $\{1, i\pi, -1, e\}$. By Schanuel's Conjecture, at least 2 of the 4 elements must be algebraically independent. Now let's assume the contradiction that $e+\pi$ is algebraic. This implies that it is a solution to a polynomial equation, when $a_0, a_1, \ldots a_n$ are rational coefficients.

$$a_n(e+\pi)^n + a_{n-1}(e+\pi)^{n-1} + \ldots a_1(e+\pi) + a_0 = 0.$$

Now expanding the terms with $b_0, b_1, \ldots b_n$ being the coefficients that we get from expanding, we are left with:

$$b_n(\pi)^n + b_{n-1}(e)^n + \ldots b_2(\pi) + b_1(e) + b_0 = 0. \tag{8.1}$$

This equation shows a polynomial that links e and π algebraically, showing that they are not algebraically independent over $\mathbb{Q}$. Now, examining the degree of transcendence, which is the maximum number of terms in a set that are algebraically independent, we have:

(1) -1 and 1 do not contribute to the degree because they can be linked through a polynomial equation ($x^2 - 1 = 0$, for example)

(2) e and $i\pi$ do not contribute to the degree because of equation 8.1, which shows e and π are linked through a polynomial.

This conclusion violates Schanuel's Conjecture and shows that e and π must be algebraically independent, to make the transcendence degree 2 (which is necessitated by the conjecture), implying that $e+\pi$ is transcendental and in turn irrational. ■

Remark 8.4. This also proves the algebraic independence of e and π, which has been an open problem for decades. In other words, whether there exists a $P(x, y)$ so that $P(e, \pi) = 0$ has been speculated by mathematicians to not be possible, but it has never been proven.

This is just one example of the many ways conjectures such as that of Schanuel's if proven, could prove to be significant advances in the field of transcendental number theory and address a wide range of unsolved problems.

9. Applications and Hypertranscendental Functions

Transcendental numbers have a multitude of applications; other than not being able to make it a page in a mathematical paper without encountering one of these numbers, they also play an essential role in real-world applications, which this section will address. The main way of studying and representing these numbers is through transcendental functions. As introduced earlier in the paper, some situations simply cannot be modeled by polynomials: you'd either need irrational coefficients or an infinite number of terms, both of which are a (usually) very large inconvenience and not a very reliable means of approximating a relationship. It is for this reason mathematicians literally created these transcendental functions: to be able to model these relationships with ease. We have already looked at a few, like exponentials, trigonometric functions, and logarithms. Other important examples are those of the error function, erf (x), and the Gamma function, $\Gamma(x)$. These, however, are even more unique, than regular transcendental functions, and are classified as Hypertranscendental [6] or transcendentally transcendental functions, which go a step further in complexity.

Definition 9.1 (Hypertranscendental Functions)**.** These equations are not solutions to any algebraic differential equation with coefficients $\in \mathbb{Z}$ and initial conditions $\in \mathbb{A}$.

The error function, denoted erf(x), represents a scaled version of the Gaussian Integral and the curve of a Normal or Gaussian distribution. This curve is used in the context of probability and statistics and is scaled up so that the area under the "bell curve" from $-\infty$ to ∞ is 1 (to represent 100% probability).

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx,$$

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx \approx 1.$$

This function is fundamental to the study of statistics and is used on an everyday basis to calculate a variety of daily-life likelihoods, which goes to show the relevance of transcendental numbers in the world. Outputs of these functions are still yet to be proved on their transcendence, and mathematicians are working on proving a relation between equation outputs of this type which could potentially increase our accuracy of probabilistic models [1].

Another non-mundane example of a transcendental function is that of the gamma function, which is used to extend the definition of a factorial to complex numbers.

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-x} dx.$$

This function also has extremely significant uses, such as being able to calculate factorials of imaginary and non-whole numbers for higher-order mathematics and related fields such as physics, but also to make probabilistic models in statistics.

Exploring these functions and their outputs gives us insight into some of the many things we do not know about transcendental number theory, and also raises the question about hypertranscendental numbers. The general consensus at the moment is that they do exist, although none have been found or proven at the moment. Finding a potential hypertranscendental number could provide invaluable insight into the field of transcendental number theory as well as our understanding of these hypertranscendental functions.

The multitude of questions that arise from this goes to show the importance of Schanuel's Conjecture once again because we can potentially use it (if verified) to prove that any algebraic input a would yield a transcendental output for $\operatorname{erf}(a)$ or $\Gamma(a)$ (provided a is not a positive rational). This could provide insight into numbers such as $\Gamma(\frac{1}{5})$, which is yet to be proven transcendental. These functions could even be the gateway to our first mathematically proven hypertranscendental number, which could have drastic indications to mathematicians as a whole. All in all, we still know very little about this field of study in a broader sense, and it will be up to time, as well as our exploration of these fascinating problems and functions, that will tell what the future holds.

10. Acknowledgements

The journey of research and writing this entire paper was difficult and took me down quite a rabbit hole. Exploring new theorems and proof methods fascinated me, but was also confusing. I don't exaggerate when I say this paper would not be possible without the help of certain individuals.

Firstly, I'd like to thank Simon Rubinstein-Salzedo and the Euler's Circle math program for guiding me with everything: from LaTeX issues to guidance on scoping my paper, Simon did a very good job not only teaching me these core ideas but also inspiring me to write my very best. Additionally, I'd like to thank my TA from this program, Abhy Devalapura from Stanford University, who helped me lay out my paper and provided guidance as to where to take the paper and help focus it. The Euler's Circle program helped me develop my interest in mathematics to a much higher degree and has really set my curiosity for the subject into overdrive. Another individual I'd like to thank is Ronny Junkins, also from Stanford University, who took his time to help me understand a myriad of technical mathematical concepts in very simple terms.

My interest in mathematics was always fueled by curiosity, and I'd like to thank Stacey Jaehnig, my calculus teacher who taught me for two years in high school for setting that curiosity into motion. I believe this interest in mathematics is what made me want to write

this paper as well as pick this particular topic to work with, due to my past experiences and astonishment with calculus and power series.

References

[1] L. C. Andrews. *Special functions of mathematics for engineers*, volume 49. Spie Press, 1998.

[2] A. Baker. Linear forms in the logarithms of algebraic numbers (iv). *Mathematika*, 15(2):204–216, 1968.

[3] J. M. Borwein, P. B. Borwein, and D. H. Bailey. Ramanujan, modular equations, and approximations to pi or how to compute one billion digits of pi. *The American Mathematical Monthly*, 96(3):201–219, 1989.

[4] T. Y. Chow. A well-motivated proof that pi is irrational, 2024.

[5] EncyclopediaOfMath. `https://encyclopediaofmath.org/wiki/Algebraic_independence`.

[6] J. Loxton and A. Van der Poorten. A class of hypertranscendental functions. *aequationes mathematicae*, 16:93–106, 1977.

[7] I. Niven. A simple proof that π is irrational. *Pi: A Source Book*, pages 276–276, 1947.

[8] R. Steinberg and R. M. Redheffer. Analytic proof of the lindemann theorem. *Pacific Journal of Mathematics*, 2(2):231–242, 1952.

[9] E. Trojovská and P. Trojovskỳ. Schanuel's conjecture and the transcendence of power towers. *Mathematics*, 9(7):717, 2021.

[10] WolframMathWorld. `https://mathworld.wolfram.com/GelfondsTheorem.html`.

[11] WolframMathWorld. `https://mathworld.wolfram.com/SchanuelsConjecture.html`.