Hausdorff Dimension and Fractal Geometry

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Preliminaries: Fractal Dimension

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Preliminaries: Fractal Dimension

Suppose the measured length of a coastline changes with the length of the measuring stick used

The fractal dimension of a coastline quantifies how the number of scaled measuring sticks required to measure the coastline changes with the scale applied to the stick

Preliminaries: Fractal Dimension

A fractal dimension is an index for characterizing fractal patterns or sets by quantifying their complexity as a ratio of the change in detail to the change in scale.

Self Similarity

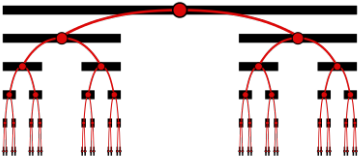


Figure: Cantor Set

A self-similar object is exactly or approximately similar to a part of itself, where the whole has the same shape as one or more of its parts. Many objects in the real world, such as coastlines, exhibit statistical self-similarity, where parts of them show the same statistical properties at many scales.

Cantor Set

The Cantor Set is an example of a self-similar object with immense importance in set theory and analysis.

- Start with the closed interval $[0, 1]$ on the real line.
- Divide it into three equal open subintervals.
- Remove the central open interval $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$:

$$[0, 1] - I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

- Repeat this process indefinitely, removing the central thirds of each remaining interval.

Definition (Cantor Set)

The Cantor set C is the intersection of all these intervals:

$$C = \bigcap_{k=0}^{\infty} C_k.$$

Scaling

Definition (Scaling Rule)

Scaling in fractal geometry describes how measurements change relative to a scaling factor ε. For a structure occupying N units at scale ε, the relationship is:

$$N = \varepsilon^{-D},$$

where D represents the fractal dimension.

Definitions

- Metric Spaces
- Exterior Measure

Metric Space

Definition (Metric Space)

A metric space is an ordered pair (M, d) where M is a set and d is a metric on M, i.e., a function:

$$d : M \times M \to \mathbb{R}$$

satisfying the following properties:

- Non-negativity: $d(x, y) \geq 0$ for all $x, y \in M$,
- Identity of indiscernibles: $d(x, y) = 0$ if and only if $x = y$,
- Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in M$,
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Exterior Measure

Definition (Exterior Measure)

If E is any subset of $\mathbb{R}^d$, the exterior measure $m^*(E)$ is defined as

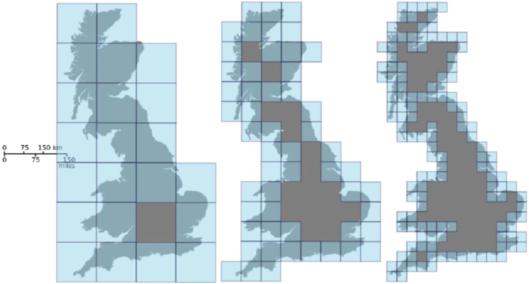
$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \mid E \subseteq \bigcup_{j=1}^{\infty} Q_j \right\},$$

where the infimum is taken over all countable coverings $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes $Q_j \subseteq \mathbb{R}^d$.

Minkowski-Bouligand Dimension

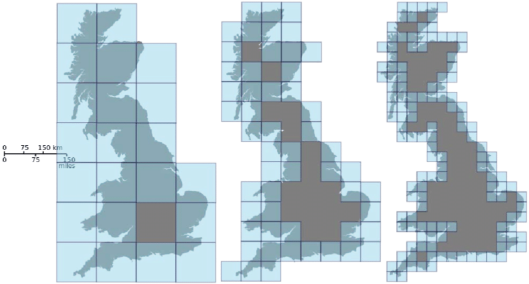
The box-counting dimension is a way of determining the fractal dimension of a set S in a Euclidean space $\mathbb{R}^n$, or more generally in a metric space (X, d).

Box-Counting Dimension



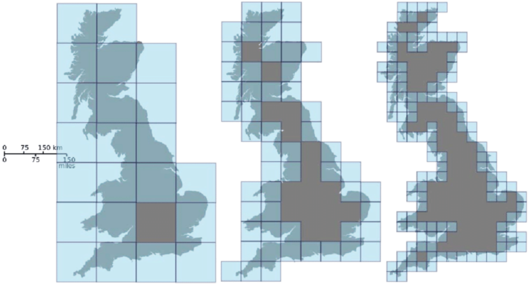
Imagine that the British coastline is placed on an evenly spaced grid.

Box-Counting Dimension



Count the number of boxes that are required to cover the set.

Box-Counting Dimension



The box-counting dimension is calculated by seeing how this number changes as we make the grid finer by applying the box-counting algorithm.

Box-Counting Dimension

Definition (Minkowski Dimension)

Suppose that $N(\epsilon)$ is the number of boxes of side length ϵ required to cover the set S. Then the box-counting dimension is defined as:

$$\dim_{\text{box}}(S) := \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log\left(\frac{1}{\epsilon}\right)}$$

Box Dimension of Sierpinski Triangle

Figure: Sierpinski Triangle

- Start with a solid closed equilateral triangle S_0 with unit sides.
- In each iteration, remove the central open triangle from each remaining triangle.
- Repeat this process indefinitely to obtain a sequence of sets S_k.

Box Dimension of the Sierpinski Triangle

- The box dimension d is given by the formula:

$$d = \lim_{k \to \infty} \frac{\log N_k}{\log \left(\frac{1}{\epsilon_k} \right)}$$

where $N_k = 3^k$ and $\epsilon_k = \left(\frac{1}{2}\right)^k$.

- Substituting N_k and ϵ_k into the formula:

$$d = \lim_{k \to \infty} \frac{\log 3^k}{\log (2^k)} = \frac{\log 3}{\log 2}$$

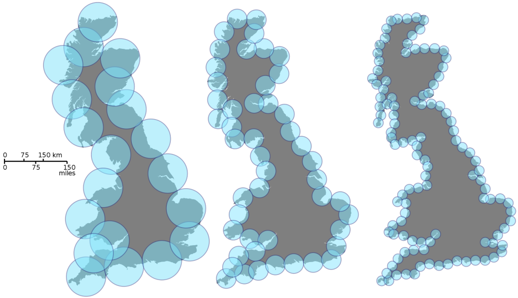
- Thus, the box dimension of the Sierpinski triangle is:

$$d = \frac{\log 3}{\log 2} \approx 1.58496$$

Hausdorff Dimension

The Hausdorff Dimension is a way of determining the fractal dimension which is similar to the Box-counting dimension. However, it is more widely regarded because of its ability to calculate the roughness of more complex and less "well-behaved" sets.

Hausdorff Dimension



The Hausdorff dimension is calculated by covering the fractal S with open balls. It is calculated as follows.

Hausdorff Measure

Definition (Hausdorff Measure)

Let (X, ρ) be a metric space. For any subset $S \subseteq X$, the Hausdorff measure $\mathcal{H}_d^\delta(S)$ is defined as:

$$\mathcal{H}_d^\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \text{diam } U_i < \delta \right\},$$

where diam U denotes the diameter of the set U:

$$\text{diam } U := \sup\{\rho(x, y) : x, y \in U\},$$

with diam $\emptyset := 0$.

Lebesgue Outer Measure and Measurability

Definition

For any interval $I = [a, b]$ or $I = (a, b)$ in $\mathbb{R}$, let $\ell(I) = b - a$ denote its length. For any subset $E \subseteq \mathbb{R}$, the **Lebesgue outer measure** $\lambda^*(E)$ is defined as:

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \right.$$

Hausdorff d-Dimensional Measure

First, an outer measure is constructed: Let X be a metric space. If $S \subset X$ and $d \in [0, \infty)$,

$$H_\delta^d(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}\, U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \text{diam}\, U_i < \delta \right\},$$

where the infimum is taken over all countable covers U of S.

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Definition (Hausdorff Dimension)

The Hausdorff d-dimensional outer measure is then defined as

$$\mathcal{H}^d(S) = \lim_{\delta \to 0} H_\delta^d(S),$$

and the restriction of this mapping to measurable sets justifies it as a measure, called the d-dimensional Hausdorff Measure.

Menger Sponge Construction

The Menger sponge M is constructed by iteratively removing smaller cubes from a larger cube, following a recursive self-similar pattern. Each face of the cube is divided into 9 smaller squares, with the central square and smaller squares removed at each iteration.

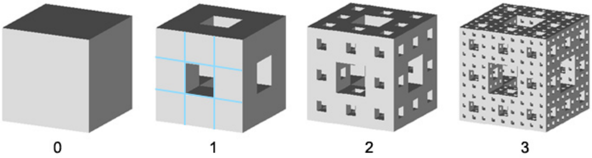

Figure: Menger Sponge

Hausdorff Dimension of the Menger Sponge

To calculate the Hausdorff dimension D_H of the Menger sponge, we use the concept of Hausdorff measure.

First Iteration:

At the first iteration, after removing central and smaller cubes:

$$\mathcal{H}^s(M_1) = \left(\frac{8}{27}\right)^s \mathcal{H}^s([0,1]^3),$$

where $\mathcal{H}^s([0,1]^3)$ is the Lebesgue measure of the unit cube in $\mathbb{R}^3$.

Recursive Definition:

For subsequent iterations, the Hausdorff measure $\mathcal{H}^s(M_k)$ is recursively defined by:

$$\mathcal{H}^s(M_k) = \left(\frac{8}{27}\right)^s \mathcal{H}^s(M_{k-1}).$$

Hausdorff Dimension

Hausdorff Dimension: The Hausdorff dimension D_H of the Menger sponge is the unique value s for which $\mathcal{H}^s(M) > 0$ and $\mathcal{H}^s(M) < \infty$:

$$D_H = \lim_{k\to\infty} \frac{\log\left(\frac{8}{27}\right)^k \mathcal{H}^s([0,1]^3)}{\log\left(\frac{1}{3}\right)^k}.$$

Simplifying,

$$D_H = \frac{\log 20}{\log 3}.$$

Therefore, the Hausdorff dimension D_H of the Menger sponge is $\frac{\log 20}{\log 3}$.

Brownian Motion

- Stochastic Process
- Random Variables

Brownian Motion

Definition (Stochastic Process)

A stochastic process $\{X(t), t \in T\}$ is a collection of random variables indexed by a parameter set T, often representing time. Each $X(t)$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Brownian Motion

Definition (Random Variables)

A random variable X is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ to the real numbers $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of events, and P is a probability measure. Specifically, for Brownian motion $B(t)$, each $B(t)$ for $t \geq 0$ is a random variable.

Hausdorff Dimension of Brownian Motion

Definition (One-dimensional Brownian Motion)

A one-dimensional Brownian motion $B(t)$ is a stochastic process defined on $[0, \infty)$ such that:

$$B(0) = 0,$$

and for any $t > 0$, $B(t)$ has:

- Independent increments,
- Normally distributed increments with mean 0 and variance t,
- Continuous paths.

Hausdorff Dimension of Brownian Motion

Let μ_B denote the measure defined by $\mu_B(A) = m(B^{-1}(A)) \cap [0,1]$, or equivalently,

$$\int_{\mathbb{R}^n} f(x)\, d\mu_B(x) = \int_0^1 f(B(t))\, dt$$

for all bounded measurable functions f. Our goal is to show that for any $0 < \alpha < 2$,

$$E[I_\alpha(\mu_B)] = E\left[\int\int \frac{1}{|x-y|^\alpha}\, d\mu_B(x)\, d\mu_B(y)\right] < \infty.$$

Hausdorff Dimension of Brownian Motion

Evaluating the expectation of increments yields:

$$E[|B(t) - B(s)|^{-\alpha}] = |t - s|^{-\alpha/2} \int_{\mathbb{R}^n} c_n |z|^n e^{-|z|^2/2} \, dz,$$

where c_n is a constant dependent on n. Simplifying,

$$E[I_\alpha(\mu_B)] \leq 2k \int_0^1 u^{-\alpha/2} \, du < \infty.$$

Thus, $I_\alpha(\mu_B) < \infty$ almost surely. By the energy method, we infer that dim Range $> \alpha$ almost surely. Letting $\alpha \to 2$ provides the lower bound on the range. Since the graph can be projected onto the range by a Lipschitz map, the graph dimension is at least the range dimension. Therefore, if $n \geq 2$, then almost surely dim Graph ≥ 2.