Unveiling Fractal Geometry through the Hausdorff Dimension

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July 14, 2024

Abstract

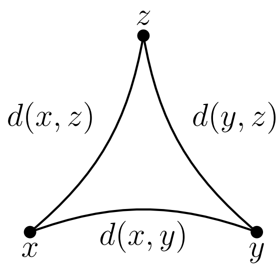
"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line"[1] Fractals, with their intricate self-similar structures, challenge traditional geometric concepts. This paper delves into the power of the Hausdorff Dimension, a tool that unveils the unique dimensional fingerprint of a fractal. By examining the complex nature of these irregular forms, we reveal hidden dimensions that defy conventional geometry.

1 Introduction

Fractal geometry is a rebellion against classical calculus. While classical calculus adeptly handles smooth curves and well-behaved functions, the discovery of fractals introduces complex shapes that defy these traditional tools. Fractals are self-similar shapes with intricate details at every scale, characterized by fractional dimensions rather than integer ones. This paper aims to quantify how to calculate the dimension of a fractal, known as the fractal dimension. We first explore the concepts of self-similarity and scaling, fundamental to understanding fractals. Next, we discuss the Minkowski-Bouligand dimension (box-counting dimension), which provides an intuitive method for calculating fractal dimensions by covering the fractal with a grid of boxes. We then move on to the Hausdorff dimension, a more rigorous approach that uses Hausdorff measure to extend the idea of length, area, and volume to non-integer dimensions.

2 Preliminaries

2.1 Metric Spaces



Definition 2.1 (Metric Space)**.** A **metric space** is an ordered pair (M, d) where M is a set and d is a metric on M, satisfying:

- Non-negativity: $d(x, y) \geq 0$ for all $x, y \in M$,
- Identity of indiscernibles: $d(x, y) = 0$ if and only if $x = y$,
- Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in M$,
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

2.2 Limit of Sets

Definition 2.2 (Limit Supremum of Sets)**.** The limit supremum or outer limit of a sequence $A_1, A_2, A_3, \ldots$ of subsets of a set X is defined as:

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcap_{n=1}^{\infty} (A_n \cup A_{n+1} \cup \cdots).$$

It consists of all points x that are in infinitely many of these sets. That is, $x \in \limsup_{n\to\infty} A_n$ if and only if there exists an infinite subsequence $A_{n_1}, A_{n_2}, \ldots$ (where $n_1 < n_2 < \cdots$) of sets that all contain x; that is, such that

$$x \in A_{n_1} \cap A_{n_2} \cap \cdots .$$

Definition 2.3 (Limit Infimum of Sets)**.** The *limit infimum* or *inner limit* of a sequence $A_1, A_2, A_3, \ldots$ of subsets of a set X is defined as:

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m = \bigcup_{n=1}^{\infty} (A_n \cap A_{n+1} \cap \cdots).$$

It consists of all points that are in all but finitely many of these sets. That is, $x \in \liminf_{n\to\infty} A_n$ if and only if there exists an index $N \in \mathbb{N}$ such that $A_N, A_{N+1}, \ldots$ all contain x; that is, such that

$$x \in A_N \cap A_{N+1} \cap \cdots .$$

Lemma 2.4. *The inner limit is always a subset of the outer limit:*

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n.$$

Proposition 2.5. If the limit supremum and limit infimum of a sequence of sets are equal, then the limit of the sequence exists and is equal to this common set.

Remark. The limit supremum and limit infimum provide important tools in measure theory and probability, particularly in defining and understanding almost sure convergence.

Corollary 2.6. *If $A_n \subseteq A_{n+1}$ for all n, then*

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

2.3 σ-algebra and Borel subsets

Definition 2.7 (σ-algebra)**.** Let X be some set. A subset $\Sigma \subseteq \mathcal{P}(X)$ is called a σ-algebra if it satisfies the following three properties:

1. $X \in \Sigma$, and X is considered to be the universal set.
2. Σ is closed under complementation: If $A \in \Sigma$, then $X \setminus A \in \Sigma$.
3. Σ is closed under countable unions: If $A_1, A_2, A_3, \ldots \in \Sigma$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Definition 2.8 (Borel Sets)**.** The Borel sets on $\mathbb{R}$ are defined as follows:

1. **Open Sets**: Every open set in $\mathbb{R}$ is a Borel set.
2. **Closed Sets**: Every closed set in $\mathbb{R}$ is a Borel set.
3. **Countable Unions**: If $\{E_n\}_{n=1}^{\infty}$ is a sequence of Borel sets, then $\bigcup_{n=1}^{\infty} E_n$ is also a Borel set.
4. **Complements**: If E is a Borel set, then $\mathbb{R} \setminus E$ is also a Borel set.

Lemma 2.9. *Every open interval $(a, b) \subseteq \mathbb{R}$ is a Borel set.*

Proof. Open intervals are open sets by definition, and by Property 1 of Borel sets, every open set is a Borel set. □

Proposition 2.10. The collection of Borel sets forms a σ-algebra on $\mathbb{R}$.

Proof. To show that the Borel sets form a σ-algebra, we need to verify that they satisfy the properties of a σ-algebra:

1. The whole space $\mathbb{R}$ and the empty set $\emptyset$ are Borel sets (trivially true from the definitions).

2. Borel sets are closed under countable unions (Property 3).

3. Borel sets are closed under complements (Property 4).

Therefore, the Borel sets indeed form a σ-algebra on $\mathbb{R}$. $\square$

Corollary 2.11. *Every closed interval* $[a, b] \subseteq \mathbb{R}$ *is a Borel set.*

Proof. Closed intervals are closed sets, and by Property 2 of Borel sets, every closed set is a Borel set. $\square$

Remark. The construction of Borel sets and their σ-algebraic properties provide a foundational structure in measure theory and analysis, essential for defining measurable functions and constructing Lebesgue measure.

2.4 Open, Closed, and Compact Sets

Definition 2.12 (Open Balls)**.** The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d \mid |y - x| < r\}.$$

A subset $E \subseteq \mathbb{R}^d$ is **open** if for every $x \in E$, there exists $r > 0$ such that $B_r(x) \subseteq E$.
A set $E \subseteq \mathbb{R}^d$ is **closed** if its complement is open.
A set $E \subseteq \mathbb{R}^d$ is **bounded** if it is contained in some ball of finite radius. A bounded set is **compact** if it is also closed.

2.5 Rectangles and Cubes

Definition 2.13 (Rectangle)**.** A (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1,] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \geq b_j$ are real numbers, $j = 1, 2, \ldots, d$. In other words, we have

$$R = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : a_j \leq x_j \leq b_j \text{for all} j = 1, 2, \ldots, d\}.$$

Remark. In the given definition, a rectangle is closed and has sides parallel to the coordinate axis. In $\mathbb{R}$, the rectangles are precisely the closed and bounded intervals, while in $\mathbb{R}^2$ they are the usual four-sided rectangles. In $\mathbb{R}^3$ they are closed parallelepipeds.

Here, the lengths of the sides of the rectangle R are $b_1 - a_1, \ldots, b_d - a_d$.

Notation 1. *The volume of the rectangle* $\mathbb{R}$ *is denoted by* $|R|$.

Definition 2.14 (Volume of Rectangle)**.**

$$|R| = (b_1 - a_1) \ldots (b_d - a_d).$$

Lemma 2.15. *When $d = 1$ the "volume" equals length, and when $d = 2$ it equals area.*

Definition 2.16 (Open Rectangle)**.** An open rectangle in $\mathbb{R}^d$ is the Cartesian product of open intervals:

$$R = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d),$$

where $a_i < b_i$ for each i.

Notation 2. *The interior of an open rectangle R is denoted as:*

$$int(R) = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d).$$

Definition 2.17 (Cube)**.** A cube in $\mathbb{R}^d$ is a rectangle Q for which the lengths of its sides are equal, i.e., $b_i - a_i = \ell$ for all i, where ℓ is the side length.

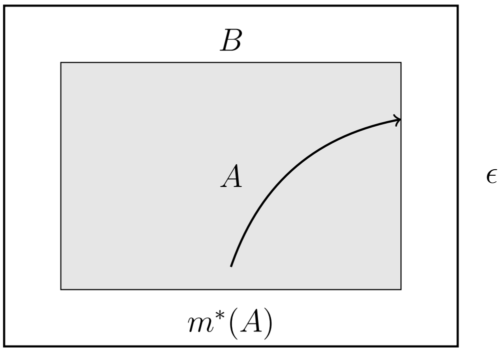
Notation 3. *The volume of Q is denoted as $|Q|$.*

Definition 2.18 (Volume of a Cube)**.** If $Q \subset \mathbb{R}^d$ is a cube with side length ℓ is:

$$|Q| = \ell^d.$$

Definition 2.19 (Almost Disjoint Union of Rectangles)**.** A union of rectangles $R_1, R_2, \ldots, R_n$ in $\mathbb{R}^d$ is said to be almost disjoint if the interiors of these rectangles are pairwise disjoint.

2.6 Exterior Measure



Definition 2.20 (Exterior Measure)**.** For any subset $E \subseteq \mathbb{R}^d$, the **exterior measure** $m^*(E)$ is defined as

$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \mid E \subseteq \bigcup_{j=1}^{\infty} Q_j \right\},$$

where the infimum is taken over all countable coverings $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes $Q_j \subseteq \mathbb{R}^d$.

Example 2.21. The exterior measure of a point is zero. This becomes evidently clear when we observe that a point is actually a cube with no volume, and which covers itself. Hence, the exterior measure of the empty set is also zero.

Example 2.22. The exterior measure of $\mathbb{R}^d$ is infinite. This follows from the fact that any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$, hence $|Q| \leq m_*(\mathbb{R}^d)$. Since Q can have arbitrary large volume, we must have $m_*(\mathbb{R}^d) = \infty$.

Observation 2.1 (Monotonicity)**.** *If $E_1 \subset E_2$, then $m^*(E_1) \leq m^*(E_2)$.*

This follows once we observe that any covering of E_2 by a countable collection of cubes is also a covering of E_1. In particular, monotonicity implies that every bounded subset of $\mathbb{R}^d$ has finite exterior measure.

Observation 2.2 (Countable sub-additivity)**.** *If $E = \bigcup_{j=1}^{\infty} E_j$, then $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$.*

Given $E = \bigcup_{j=1}^{\infty} E_j$, where E_j are subsets of $\mathbb{R}^d$, we want to show that $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$.

By the definition of exterior measure $m^*(E)$, for any $\varepsilon > 0$, there exists a countable collection of cubes $\{Q_{j,k}\}_{k=1}^{\infty}$ covering E such that

$$\sum_{k=1}^{\infty} |Q_{j,k}| \leq m^*(E_j) + \frac{\varepsilon}{2^j} \quad \text{for each } j,$$

where $|Q_{j,k}|$ denotes the volume of cube $Q_{j,k}$.

Since $E \subseteq \bigcup_{j=1}^{\infty} E_j$, any covering of E by cubes is also a covering of E_j for each j. Therefore, we have:

$$m^*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{j,k}| \leq \sum_{j=1}^{\infty} \left(m^*(E_j) + \frac{\varepsilon}{2^j} \right).$$

As $\varepsilon > 0$ was arbitrary, we can choose $\varepsilon \to 0$, yielding:

$$m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j).$$

Therefore, we have shown that $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$, which completes the proof.

Observation 2.3. *If $E \subseteq \mathbb{R}^d$, then $m^*(E) = \inf m^*(O)$, where the infimum is taken over all open sets O containing E.*

By definition, $m^*(E)$ is the exterior measure of E. For any open set $O \supseteq E$, we have $E \subseteq O$, and therefore $m^*(E) \leq m^*(O)$. This implies that $m^*(E)$ is a lower bound for $m^*(O)$ over all such open sets O.

By definition, $m^*(E)$ is the exterior measure of E. For any open set $O \supseteq E$, we have $E \subseteq O$, and therefore $m^*(E) \leq m^*(O)$. This implies that $m^*(E)$ is a lower bound for $m^*(O)$ over all such open sets O.

To show that $m^*(E)$ is the greatest lower bound, suppose $m^*(E) = \alpha$. Then for any $\epsilon > 0$, there exists a covering of E by open cubes $\{Q_j\}$ such that $\sum |Q_j| \leq \alpha + \epsilon$. Since each cube Q_j can be approximated from the inside by an open set O_j with

$$(m^*(O_j) \leq |Q_j| + \epsilon$$

Observation 2.4. *If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then $m^*(E) = m^*(E_1) + m^*(E_2)$.*

Since E_1 and E_2 are disjoint sets ($d(E_1, E_2) > 0$), any covering of E by open sets must consist of coverings of E_1 and E_2 separately. Therefore, the exterior measure of E is equal to the sum of the exterior measures of E_1 and E_2:

$$m^*(E) = \inf m^*(O) \quad \text{where } O \text{ covers } E.$$

Since $E = E_1 \cup E_2$, any such covering O can be decomposed into coverings of E_1 and E_2:

$$m^*(O) \geq m^*(E_1) + m^*(E_2).$$

Therefore, taking the infimum over all such coverings O gives us:

$$m^*(E) \geq m^*(E_1) + m^*(E_2).$$

Conversely, $E_1 \cup E_2$ itself is an open covering of E, hence:

$$m^*(E) \leq m^*(E_1) + m^*(E_2).$$

Combining both inequalities, we conclude:

$$m^*(E) = m^*(E_1) + m^*(E_2).$$

Observation 2.5. *If $E = \bigcup_{j=1}^{\infty} Q_j$, where Q_j are almost disjoint cubes, then*

$$m^*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

Since Q_j are almost disjoint cubes, the exterior measure of E is the sum of the volumes of these cubes:

$$m^*(E) = \inf m^*(O) \quad \text{where } O \text{ covers } E.$$

Any open cover O of E must cover each Q_j separately. Therefore,

$$m^*(O) \geq \sum_{j=1}^{\infty} |Q_j|.$$

Taking the infimum over all such coverings O gives us:

$$m^*(E) \geq \sum_{j=1}^{\infty} |Q_j|.$$

Conversely, $\{Q_j\}$ itself is an open covering of E, hence:

$$m^*(E) \leq \sum_{j=1}^{\infty} |Q_j|.$$

Combining both inequalities, we conclude:

$$m^*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

2.7 Lebesgue Measure

Definition 2.23 (Lebesgue Measure)**.** A subset $E \subseteq \mathbb{R}^d$ is **Lebesgue measurable** if for any $\epsilon > 0$, there exists an open set O with $E \subseteq O$ and $m_*(O - E) \leq \epsilon$. The **Lebesgue measure** $m(E)$ of a measurable set E is defined by $m(E) = m_*(E)$.

Definition 2.24 (Lebesgue Outer Measure)**.** For any subset $E \subseteq \mathbb{R}^d$, the **Lebesgue outer measure** $\lambda^*(E)$ is defined as

$$\lambda^*(E) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of open intervals with } E \subseteq \bigcup_{k=1}^{\infty} I_k\right\},$$

where $\ell(I_k)$ denotes the length of interval I_k in $\mathbb{R}$.

Property 2.1. *Every open set in* $\mathbb{R}^d ismeasurable.$

Property 2.2. *If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.*

Proof. We saw, in Observation 3 of the exterior measure, for every $\epsilon > 0$ there exists an open set O with $E \subset O$ and $m_*(O) \leq \epsilon$. Since $(O - E) \subset O$, monotonicity implies $m_*(O - E) \leq \epsilon$, as we had originally desired. □

Property 2.3. *A countable union of measurable sets is measurable*

Proof. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets. This means that each set E_n is measurable.

To show that $\bigcup_{n=1}^{\infty} E_n$ is measurable, consider any subset $A \subseteq \mathbb{R}$. We want to show that $\bigcup_{n=1}^{\infty} E_n \cap A$ is measurable.

Since each E_n is measurable, for each n, the set $E_n \cap A$ is also measurable because the intersection of a measurable set with any subset of $\mathbb{R}$ is measurable.

Now, $\bigcup_{n=1}^{\infty} E_n \cap A = \bigcup_{n=1}^{\infty}(E_n \cap A)$. This is a countable union of measurable sets (since each $E_n \cap A$ is measurable), and hence $\bigcup_{n=1}^{\infty} E_n \cap A$ is measurable.

Since $A \subseteq \mathbb{R}$ was arbitrary, we conclude that $\bigcup_{n=1}^{\infty} E_n$ is measurable. □

Property 2.4. *Closed sets are measurable.*

Proof. Let $F \subseteq \mathbb{R}$ be a closed set. We want to show that F is measurable.

Recall that a set is measurable if and only if for every subset $A \subseteq \mathbb{R}$, the intersection $F \cap A$ is measurable.

Consider any subset $A \subseteq \mathbb{R}$. Since F is closed, its complement $\mathbb{R} \setminus F$ is open. Therefore, $F \cap A = A \setminus (\mathbb{R} \setminus F)$.

Now, $A \setminus (\mathbb{R} \setminus F)$ is the difference of a set A and an open set $\mathbb{R} \setminus F$, which is measurable because the difference of a set and an open set is measurable.

Thus, $F \cap A$ is measurable for any subset $A \subseteq \mathbb{R}$.

Therefore, by definition, F is measurable.

□

Property 2.5. *The complement of a measurable set is measurable.*

Proof. Let $E \subseteq \mathbb{R}$ be a measurable set. We want to show that its complement, $\mathbb{R} \setminus E$, is measurable.

Recall that a set is measurable if and only if for every subset $A \subseteq \mathbb{R}$, the intersection $E \cap A$ is measurable.

Consider any subset $A \subseteq \mathbb{R}$. We analyze the intersection of $\mathbb{R} \setminus E$ with A:

$$(\mathbb{R} \setminus E) \cap A = A \setminus (E \cap A).$$

Since E is measurable, $E \cap A$ is also measurable. Therefore, $A \setminus (E \cap A)$ is measurable because it is the difference of the set A and the measurable set $E \cap A$.

Thus, $(\mathbb{R} \setminus E) \cap A$ is measurable for any subset $A \subseteq \mathbb{R}$.

Therefore, by definition, $\mathbb{R} \setminus E$ is measurable.

□

Property 2.6. *A countable intersection of measurable sets is measurable.*

Proof. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets. We want to show that $\bigcap_{n=1}^{\infty} E_n$ is measurable.

Recall that a set E is measurable if and only if for every subset $A \subseteq \mathbb{R}$, the intersection $E \cap A$ is measurable.

Consider any subset $A \subseteq \mathbb{R}$. We analyze the intersection of $\bigcap_{n=1}^{\infty} E_n$ with A:

$$\left(\bigcap_{n=1}^{\infty} E_n \right) \cap A = \bigcap_{n=1}^{\infty} (E_n \cap A).$$

Since each E_n is measurable, by the lemma on measurability and set operations, $E_n \cap A$ is measurable for each n. Therefore, $\bigcap_{n=1}^{\infty} (E_n \cap A)$ is also measurable as a countable intersection of measurable sets.

Thus, $(\bigcap_{n=1}^{\infty} E_n) \cap A$ is measurable for any subset $A \subseteq \mathbb{R}$.

Therefore, by definition, $\bigcap_{n=1}^{\infty} E_n$ is measurable. □

2.8 Brownian Motion

Definition 2.25 (Stochastic Process)**.** A stochastic process $\{X(t), t \in T\}$ is a collection of random variables indexed by a parameter set T, often representing time. Each $X(t)$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.26 (Random Variables)**.** A random variable X is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ to the real numbers $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of events, and P is a probability measure. Specifically, for Brownian motion $B(t)$, each $B(t)$ for $t \geq 0$ is a random variable satisfying the following properties:

- **Initial Condition:** $B(0) = 0$ almost surely.
- **Independent Increments:** For any $0 \leq t_1 < t_2 < \cdots < t_n$, the increments $B(t_2) - B(t_1), B(t_3) - B(t_2), \ldots, B(t_n) - B(t_{n-1})$ are independent random variables.
- **Normally Distributed Increments:** For any $0 \leq s < t$, the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, i.e., $B(t) - B(s) \sim \mathcal{N}(0, t - s)$.
- **Continuous Paths:** The function $t \mapsto B(t)$ is almost surely continuous.

3 Fractal Dimension

A fractal dimension is an index for characterizing fractal patterns or sets by quantifying their complexity as a ratio of the change in detail to the change in scale.

3.1 Self-similar Shapes

A self-similar object is exactly or approximately similar to a part of itself, where the whole has the same shape as one or more of its parts. Many objects in the real world, such as coastlines, exhibit statistical self-similarity, where parts of them show the same statistical properties at many scales.

3.1.1 Cantor Set

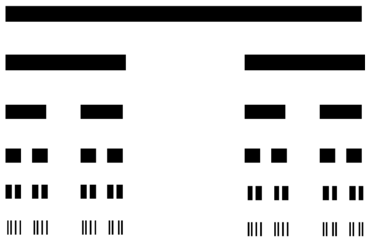


Figure 1: Cantor Set [2]

The Cantor set is of paramount importance in set theory and analysis. It can be defined and constructed in a plethora of ways. While Cantor's original definition was purely abstract, the most intuitive approach is the "middle-thirds" or ternary construction:

- Start with the closed interval $[0, 1]$ on the real line.
- Divide it into three equal open subintervals.
- Remove the central open interval $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$:

$$[0,1] - I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

- Repeat this process indefinitely, removing the central thirds of each remaining interval.

Definition 3.1 (Cantor Set)**.** The Cantor set C is the intersection of all these intervals:

$$C = \bigcap_{k=0}^{\infty} C_k.$$

Remark. A Cantor set C is measurable and has exterior measure 0.

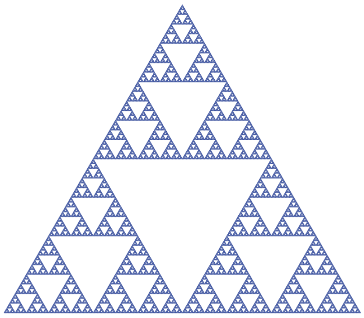


Figure 2: Sierpinski Triangle [3]

3.1.2 Sierpinski Triangle

The Sierpinski triangle is another example of a self-similar structure. It is constructed iteratively as follows:

- Start with a solid closed equilateral triangle S_0 with unit sides.
- In each iteration, remove the central open triangle from each remaining triangle.
- Repeat this process indefinitely to obtain a sequence of sets S_k.

Definition 3.2 (Sierpinski Triangle)**.** The Sierpinski triangle S is the limit of this iterative process:

$$S = \bigcap_{k=0}^{\infty} S_k.$$

3.1.3 Formal Definition of Self-Similarity

Definition 3.3 (Self-Similarity)**.** A compact topological space X is self-similar if there exists a finite set S indexing a set of non-surjective homeomorphisms $\{f_s : s \in S\}$ such that

$$X = \bigcup_{s \in S} f_s(X).$$

This structure $\mathfrak{L} = (X, S, \{f_s : s \in S\})$ defines X as self-similar within some larger space Y.

3.2 Scaling

Fractals exhibit scaling properties that differ from traditional geometric shapes, revealing complexity at all scales.

Definition 3.4 (Scaling Rule)**.** Scaling in fractal geometry describes how measurements change relative to a scaling factor ε. For a structure occupying N units at scale ε, the relationship is:

$$N = \varepsilon^{-D},$$

where D represents the fractal dimension.

Proposition 3.5 (Fractal Scaling Rule)**.** The fractal scaling rule illustrates how the number of measurement units needed to cover a fractal structure changes with the scaling factor ε. This relationship often leads to non-intuitive outcomes due to the intricate self-similarity of fractals.

Remark. For example, when measuring a fractal line initially with a stick scaled by $\frac{1}{3}$, it may unexpectedly require more sticks than expected due to the fractal's ability to reveal complexity across all scales.

4 Minkowski-Boulingand Dimension

The **Minkowski–Bouligand** or (**Box-counting**) dimension is one of the most intuitive methods of determining the fractal dimension of a set S in a Eulclidean space $\mathbb{R}^n$, or more generally in a metric space (X, d).

- Take a fractal S, for example, the Von Koch Curve.
- Divide the region containing the fractal into a grid of evenly spaced boxes of varying sizes, denoted by ϵ.
- For each box size ϵ, count the number of boxes that intersect with the fractal pattern. Denote this count as $N(\epsilon)$.

Lemma 4.1 (Box-Counting Method)**.** *The relationship between the box size ϵ and the number of boxes $N(\epsilon)$ that intersect with the fractal pattern is approximated by:*

$$N(\epsilon) \sim \epsilon^{-D},$$

where D is the fractal dimension of the pattern.

This relationship is crucial in estimating the fractal dimension D using the Box-counting algorithm. Plotting $\log N(\epsilon)$ against $\log \epsilon$ provides a practical method to estimate D from empirical data.

Definition 4.2 (Minkowski-Bouligand Dimension)**.** For a fractal S lying on a grid, suppose that $N(\varepsilon)$ is the number of boxes of side length ε required to cover the set. Then the box counting dimension is defined as :

$$dim_{box}(S) := \lim_{\epsilon \to 0} \frac{log N(\varepsilon)}{log(1/\varepsilon)}$$

Proposition 4.3 (Alternative Definition)**.** An alternative definition for the Minkowski dimension is as follows: Consider the covering number $N_{\text{covering}}(\epsilon)$ as the minimum number of open balls of radius ϵ required to cover the fractal, such that their union contains the fractal S. Additionally, define the intrinsic covering number $N'_{\text{covering}}(\epsilon)$ similarly, but with the requirement that the centers of the open balls lie within the set S.

More formally:

- $N_{\text{covering}}(\epsilon)$ is the minimum number of open balls of radius ϵ needed to cover S.
- $N'_{\text{covering}}(\epsilon)$ is the minimum number of open balls of radius ϵ needed to cover S, with the additional condition that their centers lie within S.

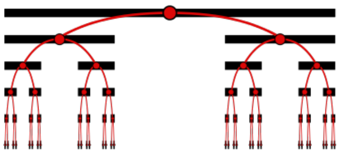


Figure 3: Cantor Set

4.1 Examples

Example 4.4 (Cantor Set)**.** The Cantor set C is constructed by repeatedly removing the middle third from each interval of the previous iteration, starting with the interval $[0, 1]$. I have already shown the construction of a Cantor Set C from $[0, 1]$ while talking about self-similarity. We know, for a general iteration we define the Cantor Set C as:

$$C = \bigcap_{k=1}^{\infty} C_k.$$

- **First Iteration:**

$$N(\varepsilon) = 2, \quad \text{where } \varepsilon = \frac{1}{3}.$$

- **Second Iteration:**

$$N(\varepsilon) = 2^2 = 4, \quad \text{where } \varepsilon = \left(\frac{1}{3}\right)^2.$$

- **General Case:** For each iteration k,

$$N(\varepsilon) = 2^k, \quad \text{where } \varepsilon = \left(\frac{1}{3}\right)^k.$$

To calculate the box dimension of the Cantor set C, we use the formula:

$$D_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $N(\varepsilon)$ is the minimum number of ε-balls needed to cover C.

Taking the limit as $\varepsilon \to 0$:

$$D_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}} = \lim_{k \to \infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}.$$

Therefore, the box dimension D_B of the Cantor set is $\frac{\log 2}{\log 3}$.

Example 4.5 (Sierpinski Triangle)**.** In the previous section on self-similirity, I have already shown the iterative construction of a Sierpinski triangle. We start with an equilateral triangle and removing the inner triangle in each iteration leads to the formation of the Sierpinski triangle S.

- **First Iteration:**

$$N(\varepsilon) = 3, \quad \text{where } \varepsilon = \frac{1}{2}.$$

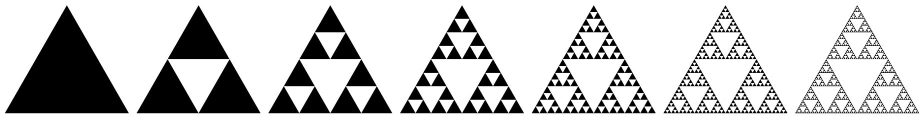


Figure 4: Sierpinski Triangle

- **Second Iteration:**

$$N(\varepsilon) = 3^2 = 9, \quad \text{where } \varepsilon = \left(\frac{1}{2}\right)^2.$$

- **General Case:** For each iteration k,

$$N(\varepsilon) = 3^k, \quad \text{where } \varepsilon = \left(\frac{1}{2}\right)^k.$$

Taking the limit as $\varepsilon \to 0$:

$$D_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}} = \lim_{k \to \infty} \frac{\log 3^k}{\log 2^k} = \frac{\log 3}{\log 2}.$$

Therefore, the box dimension D_B of the Sierpinski triangle is $\frac{\log 3}{\log 2}$.

Example 4.6 (Von Koch Curve)**.** The von Koch curve K is constructed by starting with an equilateral triangle and replacing the middle third of each line segment with two segments that form an equilateral triangle. The process is iteratively applied to each segment.

- **First Iteration:**

$$N(\varepsilon) = 4, \quad \text{where } \varepsilon = \frac{1}{3}.$$

- **Second Iteration:**

$$N(\varepsilon) = 4 \cdot 4 = 16, \quad \text{where } \varepsilon = \left(\frac{1}{3}\right)^2.$$

- **General Case:** For each iteration k,

$$N(\varepsilon) = 4^k, \quad \text{where } \varepsilon = \left(\frac{1}{3}\right)^k.$$

Taking the limit as $\varepsilon \to 0$:

$$D_B = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}} = \lim_{k \to \infty} \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}.$$

Therefore, the box dimension D_B of the von Koch curve is $\frac{\log 4}{\log 3}$.

5 Hausdorff Dimension

In this paper, we began with understanding Fractal Dimensions. We took a look at self-similar shapes, and the role of scaling for the same. Then, we saw the Minkowski-Bouligand dimension, which is widely regarded as one of the most intuitive methods to calculate a fractal dimension. However, calculating fractal dimension of a structure becomes extremely complicated when the shapes are not well-behaved. Hence, we use use the Hausdorff Dimension.

The formal definition of the Hausdorff dimension is arrived at by defining first the d-dimensional Hausdorff measure, a fractional-dimension analogue of the Lebesgue measure.

5.1 Hausdorff Measure

Definition 5.1 (Hausdorff Measure)**.** Let (X, ρ) be a metric space. For any subset $U \subset X$, let $\operatorname{diam} U$ denote its diameter:

$$\operatorname{diam} U := \sup\{\rho(x,y) : x, y \in U\}, \quad \operatorname{diam} \emptyset := 0$$

For any subset $S \subset X$ and $\delta > 0$, define:

$$H^d_\delta(S) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \operatorname{diam} U_i < \delta\right\}$$

The Hausdorff d-dimensional measure is then defined as:

$$H^d(S) = \lim_{\delta \to 0} H^d_\delta(S)$$

Property 5.1 (Monotonicity)**.** *If $E_1 \subset E_2$, then $m^*_\alpha(E_1) \leq m^*_\alpha(E_2)$.*

Proof. The Hausdorff measure is monotonic. This means that if A and B are subsets of a metric space with $A \subseteq B$, then

$$\mathcal{H}^d(A) \leq \mathcal{H}^d(B).$$

In other words, the measure of a subset cannot exceed the measure of the set containing it. □

Property 5.2 (Sub-additivity)**.**

$$m^*_\alpha\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*_\alpha(E_j)$$

for any countable family $\{E_j\}$ of sets in $\mathbb{R}^d$.

Proof. The Hausdorff measure is also sub-additive. This property states that for any countable collection of subsets $\{E_j\}$,

$$m^*_\alpha\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*_\alpha(E_j).$$

Fix $\delta > 0$, and choose for each j a cover $\{F_{j,k}\}_{k=1}^{\infty}$ of E_j by sets of diameter less than δ such that

$$\sum_k (\text{diam } F_{j,k})^\alpha \leq \mathcal{H}_\alpha^\delta(E_j) + \frac{\epsilon}{2^j}.$$

Since $\bigcup_{j,k} F_{j,k}$ is a cover of E by sets of diameter less than δ, we must have

$$\mathcal{H}_\alpha^\delta(E) \leq \sum_{j=1}^{\infty} \mathcal{H}_\alpha^\delta(E_j) + \epsilon.$$

This implies

$$\mathcal{H}_\alpha^\delta(E) \leq \sum_{j=1}^{\infty} m_\alpha^*(E_j) + \epsilon.$$

Since ϵ is arbitrary, the inequality $\mathcal{H}_\alpha^\delta(E) \leq \sum_{j=1}^{\infty} m_\alpha^*(E_j)$ holds, and we let δ tend to 0 to prove the countable sub-additivity of m_α^*. ☐

Property 5.3 (Additivity for Disjoint Sets)**.** *If $d(E_1, E_2) > 0$, then $m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$.*

Proof. If the distance between E_1 and E_2 is positive, the Hausdorff measure is additive over the union of these two sets. Specifically, if $d(E_1, E_2) > 0$, then

$$m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2).$$

This property indicates that when E_1 and E_2 are disjoint in the sense that they are positively separated, their measures add up directly. ☐

Property 5.4 (Invariance and Scaling)**.** *The Hausdorff measure is invariant under translations and rotations, and scales as follows:*

$$m_\alpha(E + h) = m_\alpha(E) \quad \textit{for all } h \in \mathbb{R}^d,$$

$$m_\alpha(rE) = m_\alpha(E) \quad \textit{where } r \textit{ is a rotation in } \mathbb{R}^d.$$

Moreover, it scales as:

$$m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E) \quad \textit{for all } \lambda > 0.$$

These properties mean that the Hausdorff measure remains unchanged when the set is translated or rotated, and it scales with a power of the scaling factor when the set is dilated.

Property 5.5. *The quantity $m_0(E)$ counts the number of points in E, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$. (Here m denotes the Lebesgue measure on $\mathbb{R}$).*

Remark. In one dimension, every set of diameter δ is contained in an interval of length δ (and for an interval, its length equals its Lebesgue measure).

In general, d-dimensional Hausdorff measure in $\mathbb{R}^d$ is, up to a constant factor, equal to Lebesgue measure.

Proof. Let $E \subset \mathbb{R}$ be a Borel set.

Since $m_0(E)$ counts the number of points in E, it represents the cardinality of E.

$m_1(E)$, the first-dimensional Hausdorff measure of E, is equivalent to the Lebesgue measure $m(E)$ for Borel sets in $\mathbb{R}$.

Therefore, Property 6 holds: $m_0(E)$ counts the number of points in E, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$. □

Property 5.6. *If E is a Borel subset of $\mathbb{R}^d$, then there exists a constant c_d depending only on the dimension d such that $\mathcal{H}^d(E) = c_d \cdot m(E)$.*

Proof. Let $E \subset \mathbb{R}^d$ be a Borel set.

By definition, $\mathcal{H}^d(E)$ denotes the d-dimensional Hausdorff measure of E, which measures the "size" of E in $\mathbb{R}^d$.

$m(E)$ denotes the Lebesgue measure of E, which measures the "volume" or "size" of E according to the Lebesgue measure in $\mathbb{R}^d$.

Since $\mathcal{H}^d(E)$ and $m(E)$ both measure the size of E in $\mathbb{R}^d$, there exists a constant c_d depending only on the dimension d such that

$$\mathcal{H}^d(E) = c_d \cdot m(E).$$

Therefore, Property 7 holds for any Borel subset $E \subset \mathbb{R}^d$. □

Property 5.7 (Property 7'). *If E is a Borel subset of $\mathbb{R}^d$ and $m(E)$ is its Lebesgue measure, then there exists a constant $c_d > 0$ such that*

$$c_d \cdot \mathcal{H}^d(E) \leq m(E) \leq 2^d \cdot c_d \cdot \mathcal{H}^d(E).$$

Proof. Let $E \subset \mathbb{R}^d$ be a Borel set and $m(E)$ its Lebesgue measure.

By definition, $\mathcal{H}^d(E)$ denotes the d-dimensional Hausdorff measure of E, which measures the "size" of E in $\mathbb{R}^d$.

Since $\mathcal{H}^d(E)$ and $m(E)$ both measure the size of E in $\mathbb{R}^d$, there exists a constant $c_d > 0$ such that

$$m(E) \leq 2^d \cdot c_d \cdot \mathcal{H}^d(E).$$

This inequality holds because $\mathcal{H}^d(E)$ represents a kind of upper bound on the "size" of E, and $m(E)$ cannot exceed 2^d times this measure.

Similarly, there exists a constant $c_d > 0$ such that

$$m(E) \geq c_d \cdot \mathcal{H}^d(E).$$

This inequality holds because $\mathcal{H}^d(E)$ provides a lower bound on the "size" of E, and $m(E)$ must be at least c_d times this measure.

Therefore, Property 7' holds for any Borel subset $E \subset \mathbb{R}^d$. □

Property 5.8 (Property 8). *If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. Also, if $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$.*

Proof. Let $E \subset \mathbb{R}^d$ be a set with $m_\alpha^*(E) < \infty$.

- **Case 1:** $\beta > \alpha$: Since $\beta > \alpha$, $\mathcal{H}^\beta(E) = 0$ because β-dimensional Hausdorff measure is more restrictive than α-dimensional. Therefore, $m_\beta^*(E) = 0$.

- **Case 2:** $\beta < \alpha$: Since $\beta < \alpha$, $\mathcal{H}^\beta(E) = \infty$ because β-dimensional Hausdorff measure is less restrictive than α-dimensional. Therefore, $m_\beta^*(E) = \infty$.

Thus, Property 8 holds for sets $E \subset \mathbb{R}^d$. □

5.2 Hausdorff d-Dimensional Measure

Now that we have understood the Hausdorff Measure, and previously, the Lebesgue Measure, we can define the Hausdorff Dimension.

First, an outer measure is constructed: Let X be a metric space. If $S \subset X$ and $d \in [0, \infty)$,

$$H_\delta^d(S) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S,\ \operatorname{diam} U_i < \delta \right\},$$

where the infimum is taken over all countable covers $\{U_i\}$ of S. The Hausdorff d-dimensional outer measure is then defined as

$$\mathcal{H}^d(S) = \lim_{\delta \to 0} H_\delta^d(S),$$

and the restriction of this mapping to measurable sets justifies it as a measure, called the **d-dimensional Hausdorff Measure.**

Definition 5.2 (Hausdorff Dimension)**.** The Hausdorff dimension $\dim_{\mathrm{H}}(X)$ of X is defined by

$$\dim_{\mathrm{H}}(X) := \inf \left\{ d \geq 0 : \mathcal{H}^d(X) = 0 \right\}.$$

This is the same as the supremum of the set of $d \in [0, \infty)$ such that the d-dimensional Hausdorff measure of X is infinite (except that when this latter set of numbers d is empty, the Hausdorff dimension is zero).

Definition 5.3 (Alternative Definition)**.** Given a Borel subset E of $\mathbb{R}^d$, the Hausdorff dimension $\mathcal{H}^d$ is defined as:

$$\mathcal{H}^d(E) = \alpha$$

where α is the unique value such that:

$$m^\beta(E) = \begin{cases} \infty & \text{if } \beta < \alpha, \\ 0 & \text{if } \alpha < \beta. \end{cases}$$

In other words, α is given by:

$$\alpha = \sup\{\beta : m^\beta(E) = \infty\} = \inf\{\beta : m^\beta(E) = 0\}.$$

This definition reflects the critical dimension at which the Hausdorff measure $m^\beta(E)$ transitions from being infinite to zero as β varies. [4]

5.3 Examples

Example 5.4 (Menger Sponge)**.** The Menger sponge M is constructed by iteratively removing smaller cubes from a larger cube, following a recursive self-similar pattern. Each face of the cube is divided into 9 smaller squares, with the central square and smaller squares removed at each iteration.

To calculate the Hausdorff dimension D_H of the Menger sponge, we use the concept of Hausdorff measure.

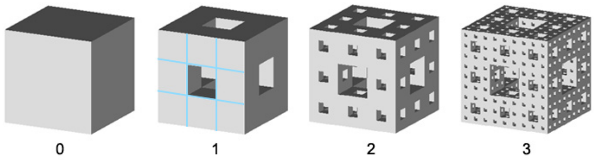


Figure 5: Menger Sponge

1. **First Iteration:**

   At the first iteration, after removing central and smaller cubes:

   $$\mathcal{H}^s(M_1) = \left(\frac{8}{9}\right)^s \mathcal{H}^s([0,1]^3),$$

   where $\mathcal{H}^s([0,1]^3)$ is the Lebesgue measure of the unit cube in $\mathbb{R}^3$.

2. **Recursive Definition:**

   For subsequent iterations, the Hausdorff measure $\mathcal{H}^s(M_k)$ is recursively defined by:

   $$\mathcal{H}^s(M_k) = \left(\frac{8}{9}\right)^s \mathcal{H}^s(M_{k-1}).$$

3. **Hausdorff Dimension:**

   The Hausdorff dimension D_H of the Menger sponge is the unique value s for which $\mathcal{H}^s(M) > 0$ and $\mathcal{H}^s(M) < \infty$:

   $$D_H = \lim_{k\to\infty} \frac{\log\left(\frac{8}{9}\right)^k \mathcal{H}^s([0,1]^3)}{\log\left(\frac{1}{3}\right)^k}.$$

   Simplifying,

   $$D_H = \frac{\log 20}{\log 3}.$$

Therefore, the Hausdorff dimension D_H of the Menger sponge is $\frac{\log 20}{\log 3}$.

Example 5.5 (Brownian Motion)**.** Brownian motion is a fundamental concept in probability theory and stochastic processes. It was first discovered by Robert Brown in 1827 while observing the erratic motion of pollen grains suspended in water. Later, Albert Einstein provided a theoretical explanation in 1905, describing it as the result of random molecular collisions.

Definition 5.6 (One-dimensional Brownian Motion)**.** A one-dimensional Brownian motion $B(t)$ is a stochastic process defined on $[0,\infty)$ such that:

$$B(0) = 0,$$

and for any $t > 0$, $B(t)$ has:

- Independent increments,

- Normally distributed increments with mean 0 and variance t,
- Continuous paths.

Brownian motion, denoted by $W(t)$, is characterized by several key properties:

Property 5.9 (Independent Increments). *For any sequence of times $0 \leq t_1 < t_2 < \cdots < t_n$, the increments $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$ are independent random variables.*

Property 5.10 (Normal Distribution). *Normal Distribution: The increments $W(t + \Delta t) - W(t)$ are normally distributed with mean 0 and variance Δt, where Δt is the time interval.*

Property 5.11 (Continuous Paths). *Brownian motion has paths that are continuous in t with probability 1.*

Theorem 5.7 (Dimensionality of Brownian Motion). *Let $\{B(t) : t \geq 0\}$ be an n-dimensional Brownian motion. If $n \geq 2$, then almost surely:*

$$dim\ Range = dim\ Graph = 2.$$

Proof. By Theorem 5.6, it suffices to establish the necessary upper bounds. Let μ_B denote the measure defined by $\mu_B(A) = m(B^{-1}(A)) \cap [0, 1]$, or equivalently,

$$\int_{\mathbb{R}^n} f(x)\, d\mu_B(x) = \int_0^1 f(B(t))\, dt$$

for all bounded measurable functions f. Our goal is to show that for any $0 < \alpha < 2$,

$$E[I_\alpha(\mu_B)] = E\left[\int\int \frac{1}{|x-y|^\alpha}\, d\mu_B(x)\, d\mu_B(y)\right] < \infty.$$

Evaluating the expectation of increments yields:

$$E[|B(t) - B(s)|^{-\alpha}] = |t-s|^{-\alpha/2} \int_{\mathbb{R}^n} c_n |z|^n e^{-|z|^2/2}\, dz,$$

where c_n is a constant dependent on n. Simplifying,

$$E[I_\alpha(\mu_B)] \leq 2k \int_0^1 u^{-\alpha/2}\, du < \infty.$$

Thus, $I_\alpha(\mu_B) < \infty$ almost surely. By the energy method, we infer that dim Range $> \alpha$ almost surely. Letting $\alpha \to 2$ provides the lower bound on the range. Since the graph can be projected onto the range by a Lipschitz map, the graph dimension is at least the range dimension. Therefore, if $n \geq 2$, then almost surely dim Graph ≥ 2. Combining with Theorem 5.6 completes the proof. [5] □

Remark. The assertion that planar Brownian motion is "almost" space filling is justified. In terms of measure, Brownian motion path behaves like a two-dimensional object. Thus, Brownian motion maximizes its spatial coverage while maintaining zero two-dimensional area.

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