Envy-Free Divisions Using Sperner's Lemma

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Fair Division

Definition

Fair division is the process of dividing a set of resources among several people who have an entitlement to them, such that each person receives their fair share.

Here are a few examples of common fair division problems:

- Cake Cutting
- Splitting a Necklace
- 3

Envy-Freeness

Definition

An envy-free division is a division such that no player believes that another participant received a share that is greater in value than their own.

Divisions are not always equal

Splitting the Rent

Problem

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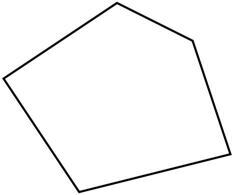
This is a popular fair division problem after it was published as an article for the New York Times because of it's use of *Sperner's Lemma*.

Definitions

A *polytope* is the convex hull of a finite number of vertices.



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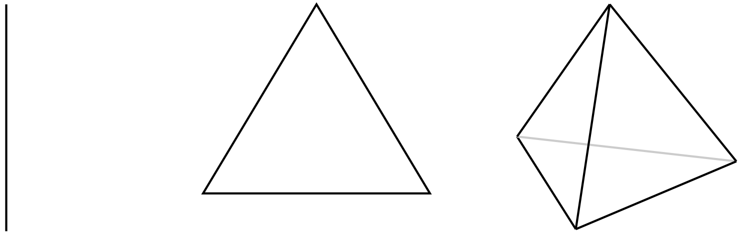
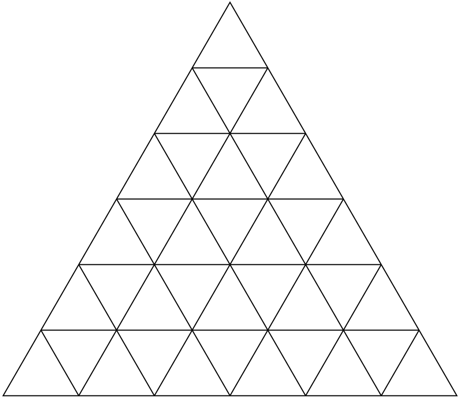


It can also be thought of as the simplest polytope in k dimensions.

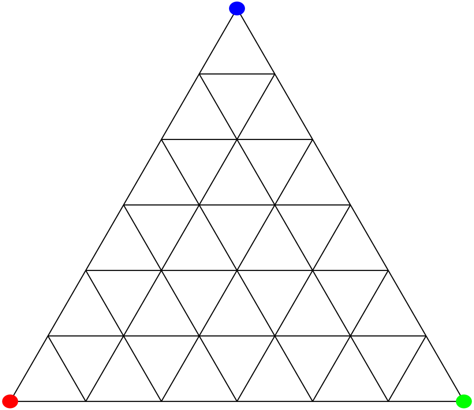
Definitions

A *triangulation* of a polytope P is any group of simplices whose union is P, and any intersection between two simplices is either empty or a shared face.



Definitions

From a *triangulation* T of P, a *Sperner labeling* can be created under the following conditions:
Each vertex of P is given a different label.



Definitions

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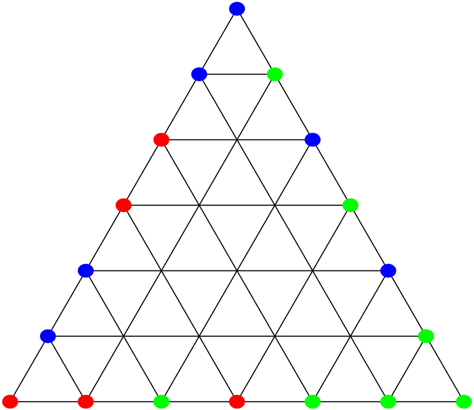
Any vertex of T which lies on a facet or side of P is given the same label as one of the vertices of P that creates the facet.



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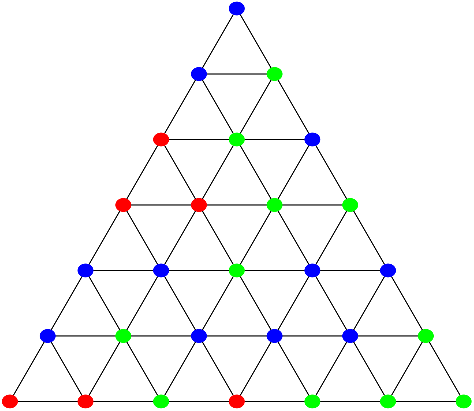
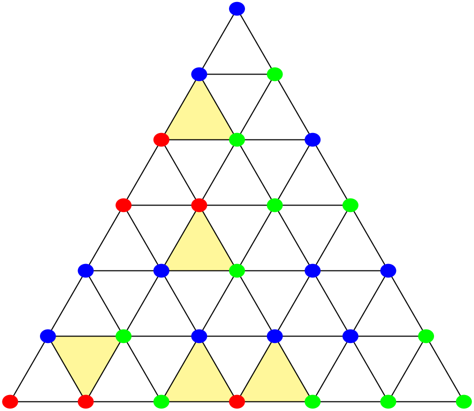


Definitions

From a *triangulation* T of P, a *Sperner labeling* can be created under the following conditions:

A *full cell* is any simplex of the same dimension as P in the triangulation T which has distinct labels for each of its vertices.



Sperner's Lemma

Lemma (Sperner's Lemma)

For any triangulation of a polytope P with a Sperner labeling, there exists at least one full cell.

Sperner's Lemma in one dimension

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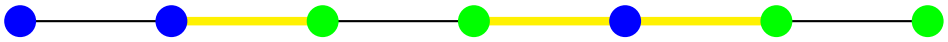

In this labeling, full cells are colored in yellow.

Full cells in one dimension can be thought of as edges where the color is changed from one to another.

Sperner's Lemma in one dimension

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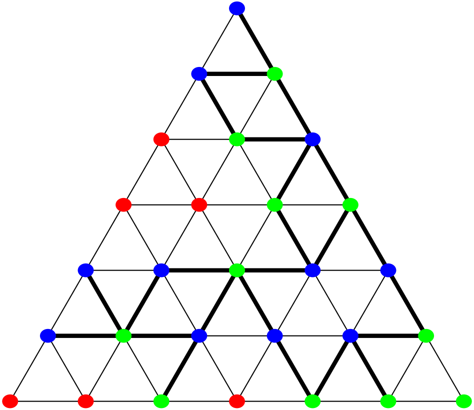
Because the triangulation T begins with one color and ends with another, there must be an odd, non-zero number of full cells.

Sperner's Lemma in two dimensions

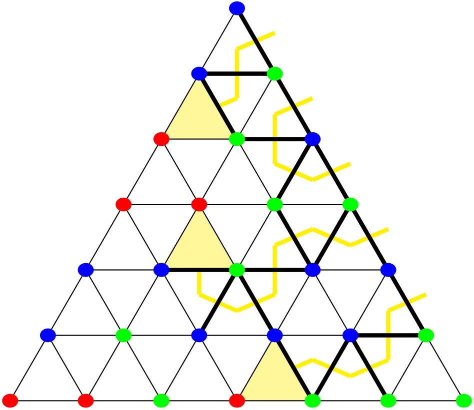
Begin with a Sperner Labeling of a triangle.



The Blue-Green edges of T can be thought of as "doors".

Sperner's Lemma in two dimensions

A path can be created by starting outside of P and traveling through doors only, leading to either a full cell or back outside the triangle.



Sperner's Lemma in two dimensions

We now can count four values:

- $\ell =$ Number of triangles with 1 green-blue edge (full cells)
- $k =$ Number of triangles with 2 green-blue edges
- $x =$ Number of green-blue edges that lie on the edge of P (odd)
- $y =$ Number of green-blue edges that lie inside P

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$$\ell + 2k = x + 2y \tag{1}$$

$$\ell = 2(y - k) + x \tag{2}$$

Proof by Induction

We have proved Sperner's Lemma in two dimensions using Sperner's Lemma in one dimension.

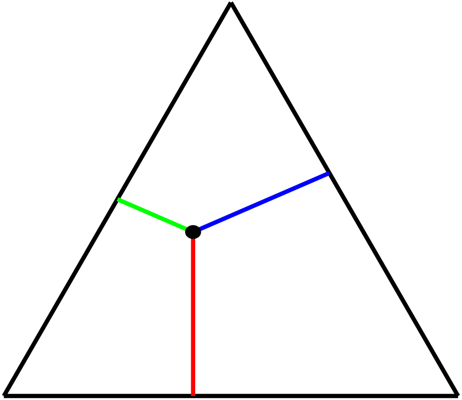
Proof by induction can be used to prove Sperner's Lemma in all dimensions.

Representing Divisions with a Triangle

Theorem (Viviani's Theorem)

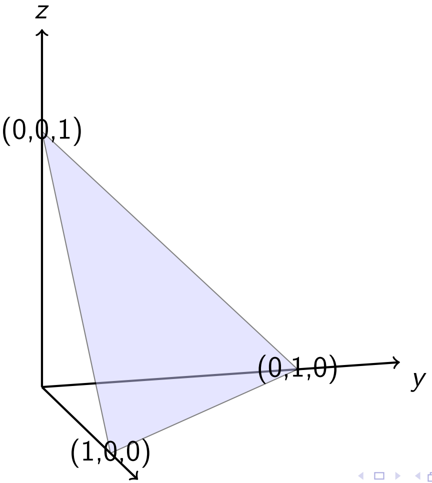
In an equilateral triangle, the sum of distance from an arbitrary point to the three sides is always constant, and equal to the height of the triangle.

Using an equilateral triangle of height 1, any point in the triangle can represent a different division of rent using it's distance from the three sides.



Representing Divisions with a Triangle

An equilateral triangle can also be created using the positive axes of the graph of $x + y + z = 1$, with the coordinates of each point representing the portion of rent paid by each room.

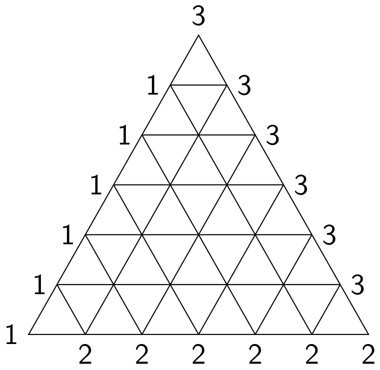


Creating a Sperner Labeling

Each point in a triangulation can be labeled as (a,b), where a is Alice's room preference and b is Bob's. Each point can then be replaced with the room neither of them chose:

Rules

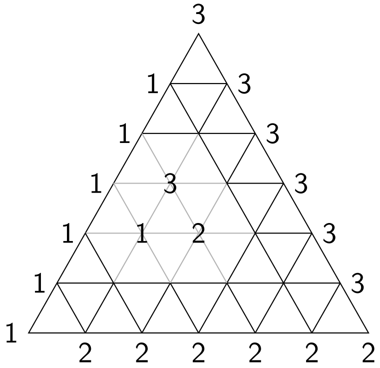
- $(1,1)$, $(1,2)$, or $(2,1)$ becomes 3
- $(2,2)$, $(2,3)$, or $(3,2)$ becomes 1
- $(3,3)$, $(3,1)$, or $(1,3)$ becomes 2



Because this is a Sperner Labeling, there must be a full cell labeled by 1, 2, and 3.

Three different numbers signify three divisions in which a different room was left unoccupied by the two players. This process can be repeated

with a triangulation containing much smaller simplices.



An Envy-Free Division

At this division of rent the third player, Carl, can choose any room and Alice and Bob will still be able to split the remaining rooms fairly.

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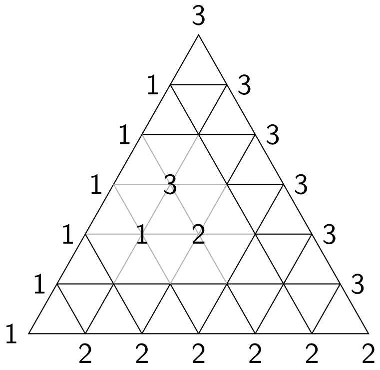

Different numbers of Players

Sperner's Lemma can be applied to multiple dimensions, making it possible to find fair divisions for more than 3 players.

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Other fair-division applications for Sperner's Lemma:

- Cake-Cutting Problems
- Division of Tasks
- Divisions of Land

Other Topological Theorems for Fair Division

The KKM Theorem and Brouwer's Fixed-Point Theorem are both proved using Sperner's Lemma. The KKM Theorem uses a covering instead of a triangulation, proving that there exists an intersection of the three colors.

Theorem (Brouwer's Fixed-Point Theorem)

For any continuous function f mapping a convex set to itself, there is a point x_0 such that $f(x_0) = x_0$.