Simple Homotopy and Simplicial homology

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Introduction

Goals: What makes two shapes the same? How can make one shape resemble another?

Simplicial Complexes

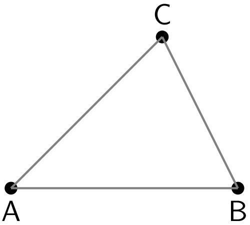
Simplicial complexes are relationships between points, edges, and higher dimensional connections.

Example

Abby and Ben have had dinner before, Ben and Claire have had dinner before, and Claire and Abby have had dinner before. However, they have never all had dinner together at the same time. This is represented in the simplicial complex as a triangle, with vertices A, B, and C.

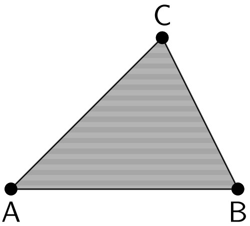
Example

Abby and Ben have had dinner before, Ben and Claire have had dinner before, and Claire and Abby have had dinner before.



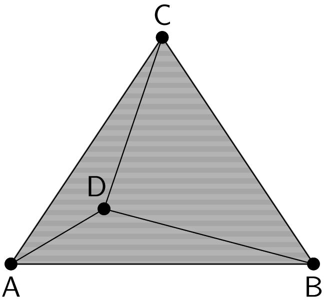
Example

Abby, Ben, and Claire decide to all have dinner together.



Example

If four people all had dinner together, the resulting simplcial complex is:



Simplical Complexes

We have a different way of representing simplicial complexes without images, and that is by using set theory.

Definition

An abstract simplicial complex K is a collection of subsets, excluding $\{\emptyset\}$, from the set $[V_n] = \{v_o, v_1, v_3, ..., v_n\}$ where n is an integer ≥ 0 such that

1. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$
2. $v_i \in K$ for every $v_i \in [v_n]$.

Simplical Complexes

This tells us that the simplicial complex K is downwards closed, and every subset of K is contained in K.

Simplical Complexes

Abby and Ben: $\{v_a, v_b\}$
Ben and Claire: $\{v_b, v_c\}$
Claire and Abby: $\{v_a, v_c\}$

i-dimensionality

Definition

A set σ of cardinality $i + 1$ is called an i-dimensional simplex

Therefore, a point is a 0-dimensional simplex, and edge is a 1-dimensional simplex, and so on.

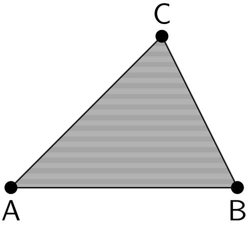
c-vector

We can represent all the dimensions of different simplices of K with a c-vector.

Definition

A c-vector is defined as $(c_0, c_1, c_2, ..., c_{dim(K)})$ where c_i is the number of simplices of dimension i, and $dim(K)$ is the dimension of the highest dimensional simplex of K.

c-vector



c-vector $= (3, 3, 1)$

codimension

$dim(\sigma) - dim(\tau)$ yields the co-dimension of τ with respect to σ.

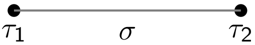
Definition

The boundary of σ in K is shown as $\delta_K(\sigma)$ and it the union of all τ such that $\{\tau \in K \text{ for all } dim(\sigma) - dim(\tau)\} = 1$

Δ^1

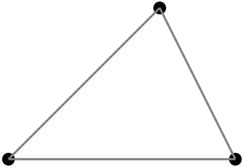
S^0

Boundary of Δ^1

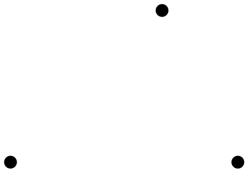


The boundary of σ in K is shown as $\delta_K(\sigma)$ and it is the union of all τ such that $\{\tau \in K \text{ for all } \dim(\sigma) - \dim(\tau)\} = 1$

S^1



Δ^2



Euler Characteristic

Definition

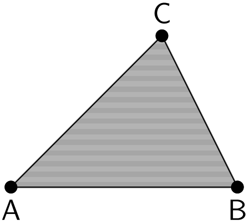
$\chi(K) = \sum_{i=0}^{n}(-1)^i c_i(K)$

Euler Characteristic

Euler Characteristic $\chi(K)$ of Simpilical Complexes. To calculate the C-Vector, we give 'weight' to points, edges, and higher dimensional simplices.

1. A point has a weight of $+1$
2. A edges has a weight of -1
3. A 'hole' created by 3 edges (for example S^1), still has a weight of -1 for each edge.
4. A 'filled in hole' has a weight of $+1$

Euler Characteristic



C-vector$= \{3, 3, 1\}$

$\chi(\Delta^2) = 3 - 3 + 1 = 1$

Simple Homotopy

Simple homotopy aims to understand what makes two 'things', in this case simplcial complexes, the same.

face-coface relations

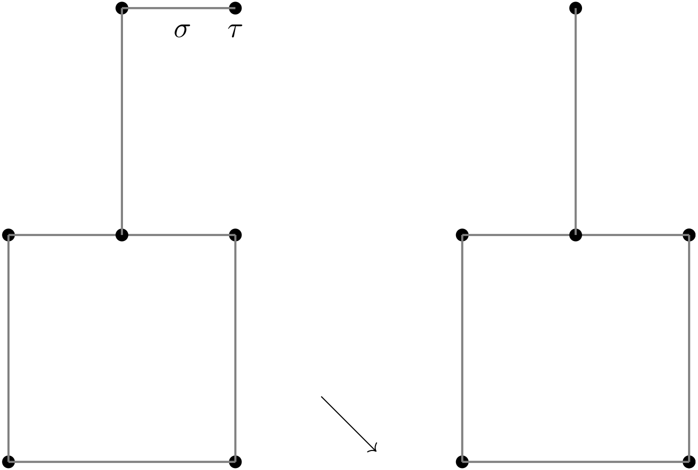
Definition

1. If in σ, $\tau \in K$ and $\tau \subseteq \sigma$, then τ is called a face of σ, and
2. σ is called the coface of τ.

Elementary Collapase

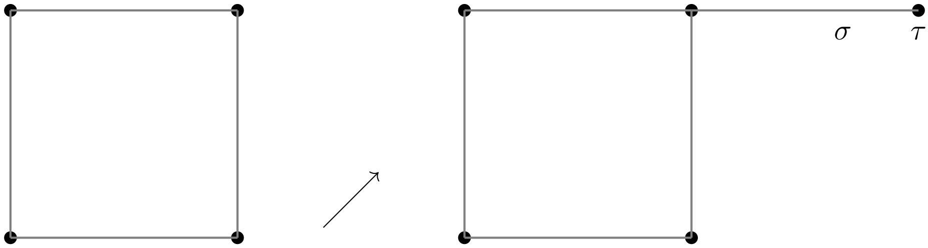
In a simplical complex K, if there is a pair of simplices σ and τ such that τ is codimension 1 in respect to σ, and τ is a face of σ and τ has no other cofaces, we can 'remove' τ and σ.

Example



Elementary Expansion

An elementary expansion of K is the union of K and $\{\tau, \sigma\}$ where $\{\tau^{(n-1)}, \sigma^{(n)}\}$ where τ is a face of σ and all other faces of σ are in K.



σ
τ

Free Pair

$\{\tau, \sigma\}$ is called free pair of K

Homotopy Type

We say that $K \sim L$ if there is a series of elementary collapses and expansions from K to L.
If $K \sim *$, K has the simple homotopy type of a point.

Proof

Proof.

Suppose $K \sim L$. Prove that $\chi(K) = \chi(L)$. We can represent the elementary collapse from K to L as a series of elementary expansions and collapses, each with adding or removing a free pair $\{\tau, \sigma\}$ such that σ and τ are codimension 1. Therefore, we know that that the Euler characteristic of $\{\tau, \sigma\}$ must be 0, as $-1 + 1 = 0$. Therefore, regardless of what $\chi(K)$ is, $\chi(K) = \chi(L)$. ■

Takeaways

1. If two simplical complexes have different Euler characteristic, there cannot be an elementary expansion between them.
2. Their Euler characteristics are the same, there may or may not be a series of collapses and expansions from one simplicial complex to another.

Simplical Homology

Aim to understand when there is an elementary collapse between to simplcial complexes

Rank and Nullity

Let $A : \mathbb{K}^n - > \mathbb{K}^m$ be a linear transformation so that A can be viewed as an mxn matrix. Then $\text{rank}(A) + \text{null}(A) = n$

Rank and Nullity

The rank of a matrix A is the number of non-zero rows when A is in row echelon form.

Boundary Operator

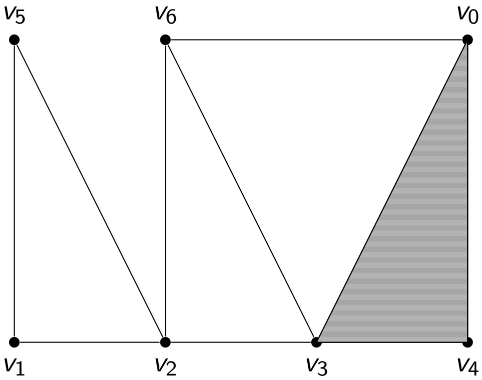
Definition

Let $\sigma \in K_m$ and write $\sigma = \sigma_{i_0}\sigma_{i_1}\cdots\sigma_{i_m}$. For $m = 0$, define $\partial_0 : k^{c_0} \to 0$ by $\partial_0 = 0$, the matrix of appropriate size consisting of all zeros. For $m \geq 1$, define the **boundary operator** $\partial_m : k^{c_m} \to k^{c_{m-1}}$ by

$$\partial_m(\sigma) := \sum_{0 \leq j \leq m} (\sigma - \{\sigma_{i_j}\}) = \sum_{0 \leq j \leq m} \sigma_{i_0}\sigma_{i_1}\cdots\hat{\sigma}_{i_j}\cdots\sigma_{i_m}$$

where $\hat{\sigma}_{i_j}$ excludes the value σ_{i_j}.

Example



$$\partial_2 = \begin{array}{c} \\ v_0v_3 \\ v_0v_4 \\ v_0v_6 \\ v_1v_2 \\ v_2v_3 \\ v_3v_4 \\ v_1v_5 \\ v_2v_5 \\ v_2v_6 \\ v_3v_6 \end{array} \begin{array}{c} v_0v_3v_4 \\ \left(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \end{array}.$$

Betti Numbers

Definition

The i-th Betti number of K is defined to be $b_i(K) = null\delta_i - rank\delta_(i+1)$

Proposition

Let K be an n-dimensional simplical complex. Supposed K can be elementary collapse to K'. Then $b_d(K) = b_d(K')$ for all $d = 0, 1, 2$

Corollary

Let $K \sim L$. Then $b_i(K) = b_i(L)$ for every integer $i \geq 0$