SIMPLICAL COMPLEXES, DISCRETE MORSE THEORY, AND SIMPLICAL HOMOLOGY

ANANYA SHAH

Date: July 15, 2024.

1. Introduction

Discrete topology and topology have vast implications in both mathematics, as well as the real world. Let's first understand these two concepts. Discrete topology is an extension of the principles of classical topology to discrete structures. It forms the backbone to analyzing complex systems through combinatorial techniques. The standard topology deals with continuous spaces while the discrete topology, on the other hand, deals with spaces that are defined by discrete sets which are endowed with what we call the discrete topology. This approach enables mathematicians to investigate connectivity, dimensionality and many other topological properties in situations where graphs and networks hold sway as predominating discrete structures. Topology serves as a means for understanding how qualitative spatial relations can be established between different geometric shapes or topological spaces represented abstractly without requiring metric information about them.

Mathematics and the discrete topology unveil the beauty of structure analysis through graphs and simplicial complexes— where simple yet discrete topological methods manage to expose mind-boggling connectivity patterns, aiding us in computing those algebraic invariants like Betti numbers, represented as B_k. These integers are as fundamental since they count k-dimensional holes or voids in a space that our simplicial complex represents.
However, despite their apparent simplicity, Betti numbers offer an insight into shape (and more importantly) connectivity which allows us to distinguish one topological entity from another. Discrete topology relies on simplicial complexes as basic structures, taking graphs to another level by higher dimensions. The composition of a simplicial complex is simplices which are vertices, edges, triangles and higher dimensional analogs arranged in the sense that each face (subset) of a simplex is included. This approach by numbers allows topological spaces to be represented and studied without having to resort to continuous concepts.

Simplicial complexes find significant use in topological data analysis (TDA). It is an innovative approach where complex datasets depicted as high-dimensional points are reimagined into simplicial complexes: allowing for an unveiling of the topological features harbored within. This technique provides a unique perspective, through TDA we are able to identify patterns and structures which would otherwise slip past the nets of traditional statistical methods; this places it as a valuable asset in fields like biology, economics, or even machine learning.
Simplicial complexes play a crucial role in computer graphics and geometry processing. Without them, it would be impossible to model or work with geometric shapes. When it comes to 3D modeling and animation, meshes are like the backbone— they consist of vertices, edges, and faces which form a simplicial complex. These techniques are like magic wands— they help in creating visually appealing images by enabling proper lighting effects through rendering or ensuring that elements do not overlap in collision detection. That's why understanding simplicial complexes is key: it allows for surface reconstruction that can lead graphical representations closer towards reality while still serving their intended functions.

Discrete Morse theory extends classical Morse theory to discrete settings, especially simplicial complexes. It provides a combinatorial approach to analyzing the topology of these complexes by studying discrete gradient vector fields. The theory identifies key points, computes homology groups including Betti numbers, and derives other topological invariants that are crucial for understanding the global structure of complex spaces.

Applications of discrete Morse theory include computer topology, which simplifies complex structures without changing their topological properties. By constructing discrete Morse functions, researchers can reduce the size of simple complex numbers, making them more manageable for computation. This simplification process is crucial for real-world applications such as image processing, sensor networks, and molecular biology, where large and complex datasets are common.
In optimization and robotics, discrete Morse theory can be used to understand topological configuration spaces, which is essential for effective path planning and resource optimization. By decomposing these spaces into simpler components, discrete Morse theory supports the development of algorithms for navigation, motion planning, and optimal resource allocation, thereby improving the efficiency and performance of complex systems.
In summary, discrete topology, simplicial complexes, and discrete Morse theory represent fundamental concepts with profound implications for mathematics and applied science. They provide a rigorous mathematical framework for analyzing complex systems and data, contributing to advances in fields ranging from data science and computer graphics to robotics and optimization. Their continued development promises to provide further insights into the underlying structures that govern our increasingly complex world.

2. Simplicial Complexes

Simplicial complexes, in quite general terms, are relationships between points, edges, and higher dimensional connections. Lets understand this with a simple analogy.

Abby and Ben have had dinner before, Ben and Claire have had dinner before, and Claire and Abby have had dinner before. However, they have never all had dinner together at the same time. This is represented in the simplicial complex as a triangle, with vertices A, B, and C. Let's say, however, that after much deliberation, Abby, Ben, and Claire decide to all have dinner together. Now there is a relationship between all three of them, and so we can represent this as a triangle, but filled in.

Extrapolating, if four people were to have the same situation, we can represent this as a tetrahedron, assuming that all four have had dinner together. Therefore we can see how simplical complexes vary across dimensions we cannot even visualize. Because of this, we have a different way of representing simplicial complexes without images, and that is by using set theory.

Definition 2.1. An abstract simplicial complex K is a collection of subsets, excluding $\{\emptyset\}$, from the set $[V_n] = \{v_o, v_1, v_3, ..., v_n\}$ where n is an integer ≥ 0 such that

(1) if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$
(2) $v_i \in K$ for every $v_i \in [v_n]$.

This tells us that the simplicial complex K is downwards closed, and every subset of K is contained in K. If we go back to our dinner analogy, we can now represent those relationships much more cleanly.

Abby and Ben having dinner together can be represented as $\{v_a, v_b\}$, Ben and Claire as $\{v_b, v_c\}$, and Claire and Abby as $\{v_a, v_c\}$. The lack of set $\{v_a, v_b, v_c\}$ alerts us that there is no relationship between all three of these points (or people). This notation allows us to represent relationships in higher dimensions that we cannot visualize.

Definition 2.2. A set σ of cardinality $i + 1$ is called an i-dimensional simplex

Therefore, a point is a 0-dimensional simplex, and edge is a 1-dimensional simplex, and so on.

We can represent all the dimensions of different simplices of K with a c-vector.

Definition 2.3. A c-vector is defined as $(c_0, c_1, c_2, ..., c_{dim(K)})$ where c_i is the number of simplices of dimension i, and $dim(K)$ is the dimension of the highest dimensional simplex of K.

Definition 2.4. (1) If in σ, $\tau \in K$ and $\tau \subseteq \sigma$, then τ is called a face of σ, and
(2) σ is called the coface of τ.

This relationship is denoted as $\tau < \sigma$; we say that τ is a face of σ.

Definition 2.5. Let σ be a simplex of simplicial complex K, with some dimension i. This is denoted as $\sigma^{(}i)$. Therefore $\tau < \sigma^i$ is any face of σ denoted as τ with a dimension strictly less than i.

If we were to take the dimension of both σ and τ and subtract them, the output of $dim(\sigma) - dim(\tau)$ yields the co-dimension of τ with respect to σ. Basically, calculating how many 'dimensions apart' both faces are. In the same vein, a boundary of σ is all the co-dimension-1 faces of σ.

Definition 2.6. The boundary of σ in K is shown as $\delta_K(\sigma)$ and it the union of all τ such that $\{\tau \in K \text{ for all } dim(\sigma) - dim(\tau)\} = 1$

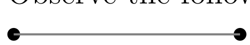
Definition 2.7. A facet of K is a simplex of K that is not contained in any other simplex of K.

Example 2.1. $\Delta^{(}n) = \mathscr{P}(\{V_n\}) - \{emptyset\}$

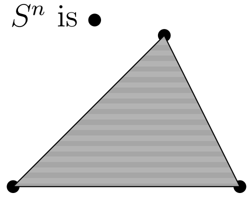
Whereas S^n is defined as

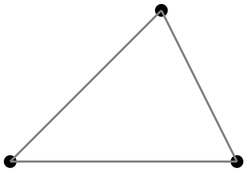
Definition 2.8. $S^n = \delta^{(}n + 1) - [V_{(}n + 1)]$

Observe the following examples. Δ^1 as



While S^n is

Δ^2 as 

And S^1 is 

Notice that for S^n, we remove the highest dimensional simplex of $\Delta^(n + 1)$

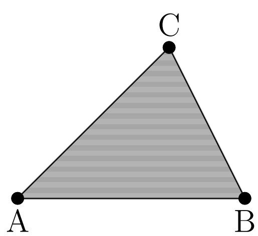
Euler Characteristic Recall the C-vector from definition 2.3. We will use the C-vector to calculate the Euler Characteristic ($\chi(K)$) of Simplical Complexes. To calculate the C-Vector, we give 'weight' to points, edges, and higher dimensional simplices.

(1) A point has a weight of $+1$
(2) A edges has a weight of -1
(3) A 'hole' created by 3 edges (for example S^1), still has a weight of -1 for each edge.
(4) A 'filled in hole' has a weight of $+1$

Definition 2.9. Euler Characteristic $\chi(K) = \sum_{i=0}^{n} (-1)^i c_i(K)$

Euler Characteristic $\chi(K)$ of Simpilical Complexes. To calculate the C-Vector, we give 'weight' to points, edges, and higher dimensional simplices.

(1) A point has a weight of $+1$
(2) A edges has a weight of -1
(3) A 'hole' created by 3 edges (for example S^1), still has a weight of -1 for each edge.
(4) A 'filled in hole' has a weight of $+1$



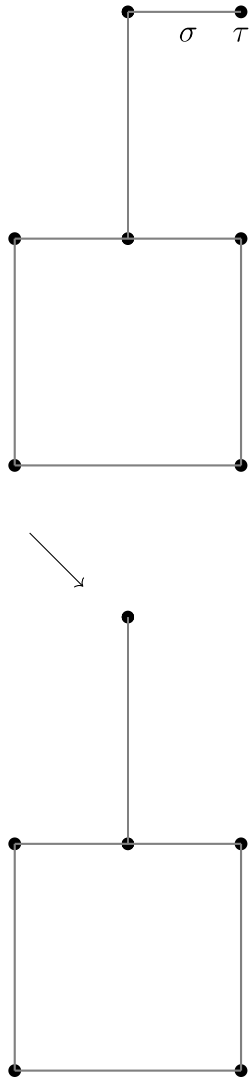
C-vector= $\{3, 3, 1\}$ $\chi(\Delta^2) = 3 - 2 + 1 = 2$

The c-vector gives us all of these values, and so we can quickly calculate the Euler characteristic of all simplicial complexes by doing an alternating sum on the C Vector. The use for a C-vector will be known shortly.

3. Simple Homotopy

Simple homotopy aims to understand what makes two 'things', in this case simplcial complexes, the same. Elementary Collapase

In a simplical complex K, if there is a pair of simplices σ and τ such that τ is codimension 1 in respect to σ, and τ is a face of σ and τ has no other cofaces, we can 'remove' τ and σ.



An elementary expansion of K is the union of K and $\{\tau, \sigma\}$ where $\{\tau^(n-1), \sigma^(n)\}$ where τ is a face of σ and all other faces of σ is in K. We say that $K \sim L$ if there is a series of elementary collapses and expansions from K to L. If $K \sim *$, K has the simple homotopy type of a point.
τ, σ is called free pair of K

Proposition 3.1. *If $K \sim L$. Then $\chi(K) = \chi(L)$*

Proof. Suppose $K \sim L$. Prove that $\chi(K) = \chi(L)$. We can represent the elementary collapse from K to L as a series of elementary expansions and collapses, each with adding or removing a free pair $\{\tau, \sigma\}$ such that σ and τ are codimension 1. Therefore, we know that that the Euler characteristic of $\{\tau, \sigma\}$ must be 0, as $-1 + 1 = 0$. Therefore, regardless of what $\chi(K)$ is, $\chi(K) = \chi(L)$. ■

Some conclusions can be made from this proof. If two simplical complexes have different Euler characteristic, there cannot be an elementary expansion between them. However, if their Euler characteristics are the same, there may or may not be a series of collapses and expansions from one simplicial complex to another. We will not know for sure until later in this paper.

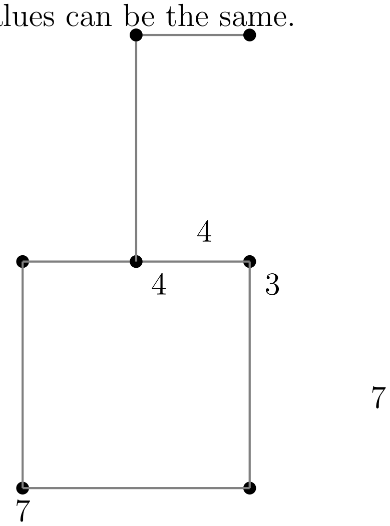
4. Discrete Morse Theory

In the last section, we introduced the concept of a hole, even thought it is still unclear what a hole is. Discrete Morse theory will be useful in defining what a hole is, and understand how to 'label' different (or the same) simplical complexes.

Definition 4.1. Like K be a simplical complex. A function $f : K \to \mathbb{R}$ is weakly increasing if $f(\sigma) \leq f(\tau)$ when σ is a face of τ.

A basic discrete morse function is a weakly increasing function that is at most 2 to 1, and when $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$, $f(\sigma) = f(\tau)$.

What this means is that at most, a number in a discrete morse function can only be used twice (therefore 2-1), and face coface pairs must have the same number, or be weakly increasing. Weakly increasing means that a higher dimensional simplex has a greater value than a lower dimensional coface. Because the function is not strictly increasing, the two values can be the same.

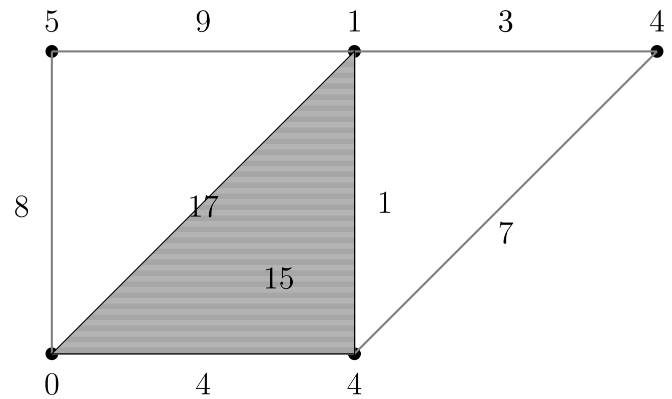


You may notice that 3 only shows up once in the basic discrete Morse function. In a basic discrete Morse function, a value that only shows up once is a critcal value. Otherwise, it is a regular value. The simplex that has the critical value is called the critical simplex.

Definition 4.2. A discrete Morse function $fonK$ is a function $f : K \to \mathbb{R}$ that for every p-simplex on K

(1) $|\tau^{(}p-1) < \sigma : f(\tau) \geq f(\sigma| \leq 1$
(2) $|\tau^{(}p+1) > \sigma : f(\tau) \leq f(\sigma| \leq 1$

This definition basically means that any higher-dimensional simplicies must have a higher or equal value to lower dimensional simplicies. However, there is one exception allowed. This exception occurs when a higher dimensional coface has a value that is less than or equal to a lower dimensional face. This an occur once or none for a face-coface pair. Lets understand this through a diagram.



We can see a couple of exceptions in this discrete Morse function. For example, value of the 2-simplex with vertices $0-4-1$ is 15, yet the coface of it (namely the edge with vertices $0-1$, has a value of 17. The edge is lower dimensional than the face, but still has a larger value. Even so, the edge does not violate any other relations, as it has a larger value than both vertices it is a coface of. Therefore, it 'uses' its on exception, and is still a discrete Morse function. A discrete Morse function has no requirements to be $2-1$, unlike a basic discrete Morse function. There are three simplcies with a value of 4 in this discrete Morse function, but it doesn't matter, as they satisfy the requirements of a discrete Morse function.

A discrete Morse function, similar to a basic discrete Morse function, wil still have critical and regular values. A critcal value in a discrete Morse function is one that no exceptions, and is represented as such.

Definition 4.3. A p-simplex σ of K is said to be critical in respect to a discrete Morse function $f : K$ if and only if

$$|\tau^[(p-1)] < \sigma : f(\tau) \geq f(\sigma)| = 0$$
$$|\tau^[(p+1)] > \sigma : f(\tau) \leq f(\sigma)| = 0$$

As shown, critical simplicies are those simplicies which do not have any exceptions. Looking back at our diagram, the critical simplices are the egde with a value of 7, the edge with a value of 4, the edge with a value of 8, the edge with a value of 9, and so on. As with basic discrete Morse functions, a simplex that is not critical is regular.

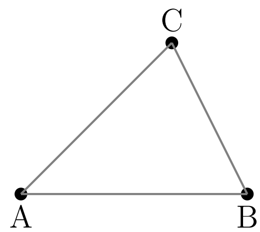
Lemma 4.4. *Exclusion lemma: Let $f : K \to \mathbb{R}$ be discrete Morse function and $\sigma \in K$ is a regular simplex.*

This tells us that if a particular simplex has an exception, any face of that simplex has also used up that exception. Discrete Morse Functions are used in persistant homology, and seeing how simplicial complexes change - or don't change - over time. However, in this paper, we will only be going into simplicial homology, which will tell us characteristics about simplicial complexes, in a general sense.

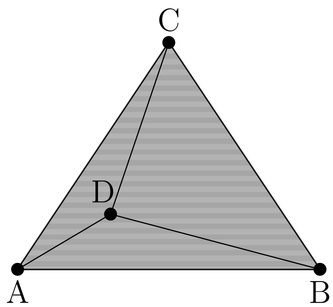
5. Simplicial Homology and the Betti Numbers

As mentioned with the Eular Characterisitc, we can determine when two simplicial complexes K is not elementary collapsable to L, via different Euler Characteristics. However, when two simplicial complexes have the same Euler characteristics, we cannot be sure that there is a elementary collapse mapping one to another. This is where simplicial homology comes in.

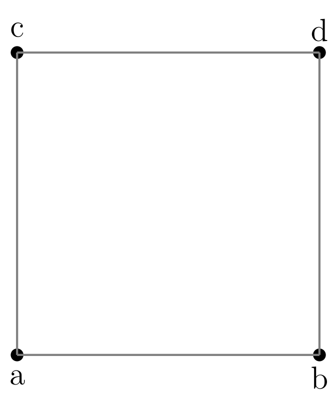
Homology aims to count the number of holes in a simplicial complex - whether it is a two-dimensional hole:



Or a three-dimensional void:



When the center of this simplicial complex is hollow. To calculate the homology of a simplicial complex, we use the Betti numbers. In order to calculate the Betti numbers, we use the rank and nullity of Simplicial Complexes by ordering them in a matrix. Let's first understand how to generate a vector space from our simplicial complexes.



We could represent the 'hole' we have here as ab, bc, cd, ad but that would open up all the sequences of orderings of the above simplicial complexes. Inevitably, this will give us sequences that repeat, or htose that are not even simplicial complexes. Instead, we would introduce the vector space of the simplicial complex, represented as the sum $ab + bc + cd + ad$, and apply modulo 2 to our vector space. Therefore, the vector space for the above simplicial complex is:
$0, ab, bc, cd, ad, ab + bc, ad + cd, ab + ad, bc + cd, bc + ad, cd + ad, ab + bc + cd, ab + bc + ad, ab + cd + ad, bc$

In order to actually put these values into a matrix, we first need to understand rank and nullity.

Rank and Nullity

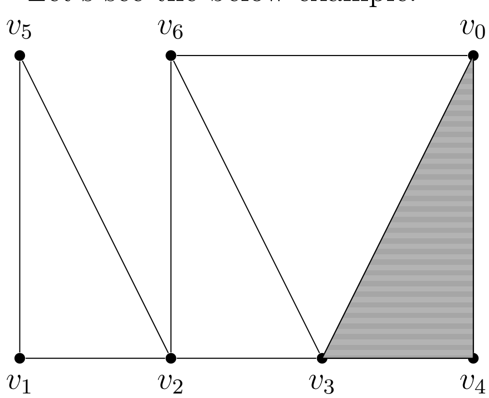
Definition 5.1. Let $A : \mathbb{K}^n -> \mathbb{K}^m$ be a linear transformation so that A can be viewed as an mxn matrix. Then rank(A) + null(A) = n

The rank of a matrix A is the number of non-zero rows when A is in row echelon form. This is relevant because as we will soon see, the nullity will count all the potential holes in a given dimension, and will be calculated with the above theorem. In a given matrix, that has been reduced with 'Row echelon form' which means that

(1) All non-zero rows are above any row of zeros
(2) The leadering coeffcient of a non-zero row is always strictly to the right of the leadering coefficient of the row above it.

The rank of a matrix A is the number of non-zero rows. It is important to point out that when row operations are performed on a martix, it does not change the rank of A.

Let's see the below example.



Now, we can find the vector spaces for each dimension of this simplical complex. $K = v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_0v_3, v_0v_4, v_0v_6, v_1v_2, v_3v_4, v_1v_5, v_2v_5, v_2v_6, v_3v_6, v_0v_3v_4$ and we can can represent all the dimensions of K like this. $K_0 = v_0, v_1, v_2, v_3, v_4, v_5, v_6$ $K_1 = v_0v_3, v_0v_4, v_0v_6, v_1v_2, v_3v_4, v_1v_5, v$ $K_2 = v_0v_3v_4$ and everything further on will be $\emptyset$ Now we are going to use a boundary operator, defined as such:

Definition 5.2. Let $\sigma \in K_m$ and write $\sigma = \sigma_{i_0}\sigma_{i_1}\cdots\sigma_{i_m}$. For $m = 0$, define $\partial_0 : k^{c_0} \to 0$ by $\partial_0 = 0$, the matrix of appropriate size consisting of all zeros. For $m \geq 1$, define the **boundary operator** $\partial_m : k^{c_m} \to k^{c_{m-1}}$ by

$$\partial_m(\sigma) := \sum_{0\leq j\leq m} (\sigma - \{\sigma_{i_j}\}) = \sum_{0\leq j\leq m} \sigma_{i_0}\sigma_{i_1}\cdots\hat{\sigma}_{i_j}\cdots\sigma_{i_m}$$

where $\hat{\sigma}_{i_j}$ excludes the value σ_{i_j}.

We had previously seen a boundary operator as all codimesion-1 faces of σ, and that intuition is how we calcuate the boundary operator for this simplicial complex. This can be represented in the below martix.

	v_0v_3	v_0v_4	v_0v_6	v_1v_2	v_2v_3	v_3v_4	v_1v_5	v_2v_5	v_2v_6	v_3v_6
v_0	1	1	1	0	0	0	0	0	0	0
v_1	0	0	0	1	0	0	1	0	0	0
v_2	0	0	0	1	1	0	0	1	1	0
v_3	1	0	0	0	1	1	0	0	0	1
v_4	0	1	0	0	0	1	0	0	0	0
v_5	0	0	0	0	0	0	1	1	0	0
v_6	0	0	1	0	0	0	0	0	1	1

As you can see, we play 1 in the matrix when the lower dimensional simplex is a face of the higher dimensional one. We can do this for $1 - simplces$ and $2 - simplicies$ as well

$\partial_2 =$

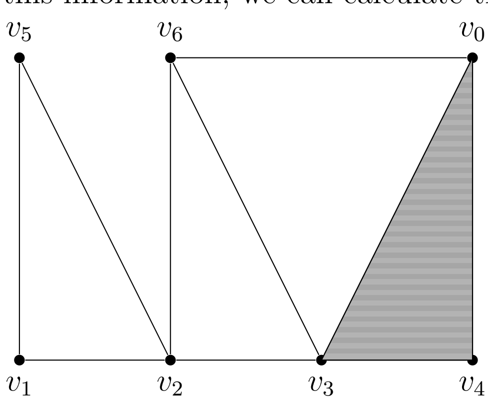
	$v_0v_3v_4$
v_0v_3	1
v_0v_4	1
v_0v_6	0
v_1v_2	0
v_2v_3	0
v_3v_4	1
v_1v_5	0
v_2v_5	0
v_2v_6	0
v_3v_6	0

So, what can we do with this information? As we saw before, we can calculate the rank and nullity of these matrices. If we were to row - reduce δ_1, we would get the rank as 6, and to calculate the nullity, we take the rank minus the number of columns, which in the cares of $\delta_!$, is 10. Therefore the nullity is $10 - 6 = 4$ Similarly, when we row reduce δ_2, the rank is =1, and therefore the nullity has to be 0. Now, we are able to calculate the Betti numbers.

Betti Numbers

Definition 5.3. The i-th Betti number of K is defined to be $b_i(K) = null\delta_i - rank\delta_(i+1)$

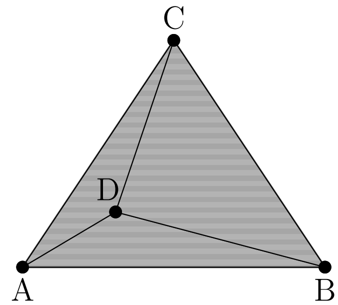
Given this information, we can calculate the Betti number of our example simplical com-



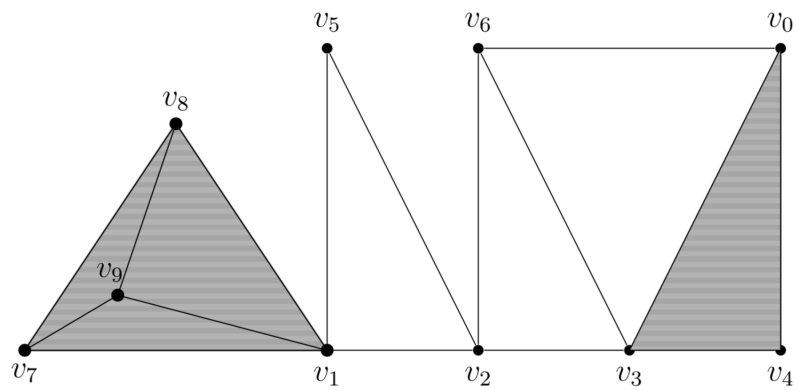
plex, K

H_0 is therefore $\daleth(7 - 6)$, which will be 1. Therefore, the 0-th Betti number, or $b_0 = 1$

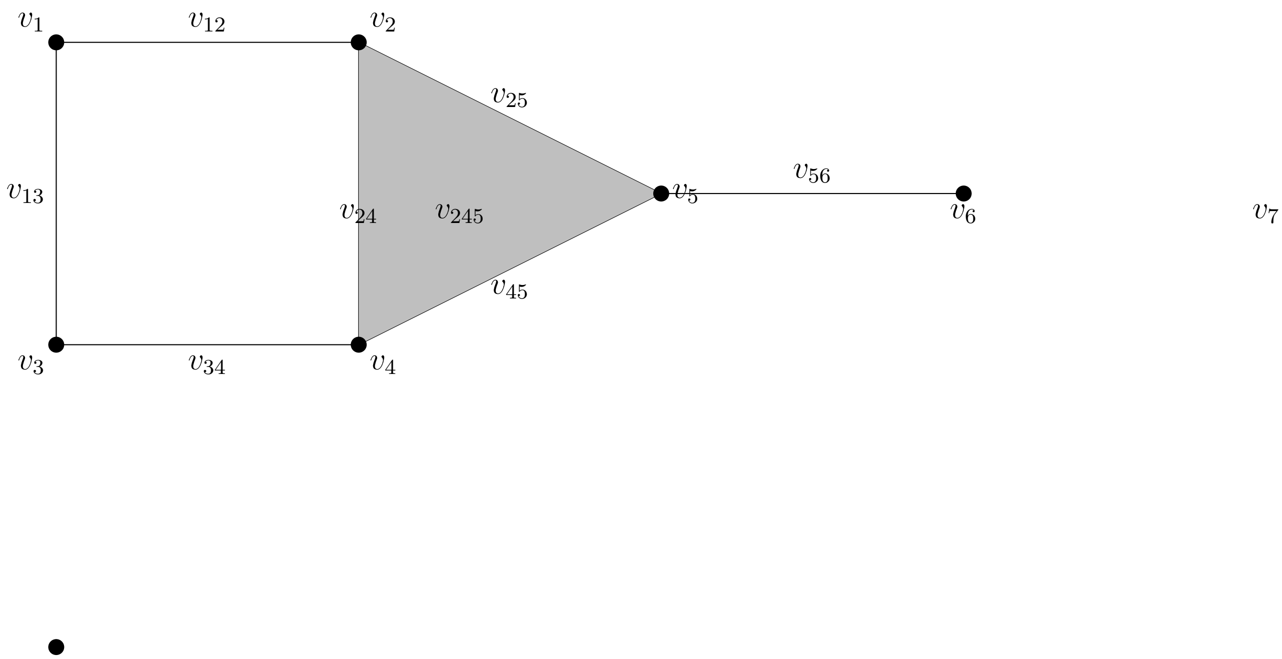
Following the same logic, $b_1 == 3$, and any larger Betti number will be zero. // But what do all of these values mean? The 0-th Betti number is the number of connected components of the simplicial complex K. Just looking at it, we know that it is going to be 1. The second Betti number is the number of holes, and again looking at the diagram, we can see that the number of 'holes' is going to be 3. Any larger Betti numbers are holes in a higher dimensional simplex. For example,



When the inside is hollow, is a void. If we had the simplicial complex



Then the 3rd Betti Number would be 1, as there is one void present. As the dimensions get higher and higher, we cannot visualise the 'holes', but the Betti numbers can tell us about datasets that we cannot visualize. Lets go through another example.


v_1
v_{12}
v_2
v_{25}
v_{13}
v_{24}
v_{245}
v_5
v_{56}
v_6
v_7
v_{45}
v_3
v_{34}
v_4

Just by looking at it, we can already get some Betti numbers. $K_2 = v_245$ $K_1 = v_12, v_13, v_24, v_34, v_25, v_45, v_56$ $K_0 = v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and $b_0 = 2$, because we have 2 components. $b_1 = 1$, because of the one hole we have, and $b_2 = 0$ and any $i > 2$ will be 0, as we have no more higher dimensional holes.

$$\partial_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and

$$\partial_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

If we where to go and reduce the matrices, δ_1 would reduce to

$$\delta_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ending with 5 non-zero rows, and our nullity being the columns minus the rows, we get $\text{null}(\delta_1) = 7 - 5 = 2$. The rank of $delta_2$ is 1, and so the nullity is $1 - 1 = 0$, and the rank of $\delta_0 = 7$, and the rank is 0. Therefore, we can conclude that:
$H_0(K) = k^{7-5} = k^2$
$H_1(K) = k^{2-1} = k^1$
$H_2(K) = k^0 = 0$
Which all check out with our original observation. This leads us to the below proposition and corollary, which cannot be proved without linear algebra, and therefore will not be covered in this paper.

Proposition 5.4. *Let K be an n-dimensional simplical complex. Supposed K can be elementary collapse to K'. Then $b_d(K) = b_d(K')$ for all $d = 0, 1, 2$*

Corollary 5.5. *Let $K \sim L$. Then $b_i(K) = b_i(L)$ for every integer $i \geq 0$*

References

@book book, AUTHOR = Scoville, Nicholas A. , TITLE = Discrete Morse Theory, EDITION = First, PUBLISHER = American Mathematical Society, ISBN = 978-1-4704-5298-8, URL = www.ams.org/bookpages/stml-90,