An Exposition to the Fourier Series

Amr Nazir Ahmad

July 15, 2024

Table of Contents

Introduction

- Fourier's Claim: any function can be expanded in a series of sines and cosines of multiples of the variable (needs additional corrections)

Introduction

- Fourier's Claim: any function can be expanded in a series of sines and cosines of multiples of the variable (needs additional corrections)
- Periodic functions

Introduction

- Fourier's Claim: any function can be expanded in a series of sines and cosines of multiples of the variable (needs additional corrections)
- Periodic functions

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad (1)$$

- a_n, a_0 and b_n are called the Fourier Coefficients

Orthogonality of Functions

- Just like vectors, functions can be orthogonal

Orthogonality of Functions

- Just like vectors, functions can be orthogonal
- Their inner product has to be 0

Orthogonality of Functions

- Just like vectors, functions can be orthogonal
- Their inner product has to be 0

Definition

Inner Product of 2 functions f and g over an interval $[a, b]$ is defined as:

$$(f, g) = \int_a^b f(x)\, g(x)\, dx$$

Orthogonality of Functions

- Just like vectors, functions can be orthogonal
- Their inner product has to be 0

Definition

Inner Product of 2 functions f and g over an interval $[a, b]$ is defined as:

$$(f, g) = \int_a^b f(x)\, g(x)\, dx$$

- Then two functions f and g are orthogonal when

Condition for Orthogonality

$$(f, g) = \int_a^b f(x)\, g(x)\, dx = 0$$

Orthogonality of Trig. Functions

- All sines and cosines are orthogonal to each other

Orthogonality of Trig. Functions

- All sines and cosines are orthogonal to each other
- Two cosine functions $\cos(nx)$ and $\cos(mx)$ are orthogonal to each other except when $n = m$

Orthogonality of Trig. Functions

- All sines and cosines are orthogonal to each other
- Two cosine functions $\cos(nx)$ and $\cos(mx)$ are orthogonal to each other except when $n = m$
- Two sine functions $\sin(nx)$ and $\sin(mx)$ are orthogonal to each other except when $n = m$

Computing Fourier Coefficients

- To compute a_n, multiply both sides of Equation 1 by $\cos(kx)$, where k is some integer

Computing Fourier Coefficients

- To compute a_n, multiply both sides of Equation 1 by $\cos(kx)$, where k is some integer
- Integrate both sides from $-\pi$ to π

Computing Fourier Coefficients

- To compute a_n, multiply both sides of Equation 1 by $\cos(kx)$, where k is some integer
- Integrate both sides from $-\pi$ to π

Computing a_n

Since all the sine terms are orthogonal to $\cos(kx)$, and all cosine terms except $\cos(kx)$ are orthogonal to $\cos(kx)$,

Computing Fourier Coefficients

- To compute a_n, multiply both sides of Equation 1 by $\cos(kx)$, where k is some integer
- Integrate both sides from $-\pi$ to π

Computing a_n

Since all the sine terms are orthogonal to $\cos(kx)$, and all cosine terms except $\cos(kx)$ are orthogonal to $\cos(kx)$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx &= a_k \int_{-\pi}^{\pi} (\cos(kx))^2\,dx \\ &= a_k \pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx\end{aligned}$$

- Using similar methods, we obtain b_n; we can also compute a_0, just a special case of a_n,

Computing Fourier Coefficients

- To compute a_n, multiply both sides of Equation 1 by $\cos(kx)$, where k is some integer
- Integrate both sides from $-\pi$ to π

Computing a_n

Since all the sine terms are orthogonal to $\cos(kx)$, and all cosine terms except $\cos(kx)$ are orthogonal to $\cos(kx)$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx &= a_k \int_{-\pi}^{\pi} (\cos(kx))^2\,dx \\ &= a_k \pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx\end{aligned}$$

- Using similar methods, we obtain b_n; we can also compute a_0, just a special case of a_n,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Complex Form, Partial Sums

- We can write the Fourier Series in complex form, using Euler's formulas to substitute complex terms in place of sines and cosines

Complex Form, Partial Sums

- We can write the Fourier Series in complex form, using Euler's formulas to substitute complex terms in place of sines and cosines

Complex Form of the Fourier Series

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n \, e^{inx} \\
c_n &= \frac{a_n - ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx
\end{aligned} \tag{2}
$$

Complex Form, Partial Sums

- We can write the Fourier Series in complex form, using Euler's formulas to substitute complex terms in place of sines and cosines

Complex Form of the Fourier Series

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n \, e^{inx} \\
c_n &= \frac{a_n - ib_n}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx
\end{aligned} \tag{2}
$$

- We will need to work now with the partial sums of the Fourier Series

Complex Form, Partial Sums

- We can write the Fourier Series in complex form, using Euler's formulas to substitute complex terms in place of sines and cosines

Complex Form of the Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$
$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \tag{2}$$

- We will need to work now with the partial sums of the Fourier Series

Definition

The N-th partial sum of the Fourier Series is defined as

$$S_N(f,x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{3}$$

Dirichlet Kernel

- A Kernel is the set of elements that goes to 0 under a transformation

Dirichlet Kernel

- A Kernel is the set of elements that goes to 0 under a transformation

Definition

The N-th Dirichlet Kernel is a collection of 2π periodic functions is defined as,

Dirichlet Kernel

- A Kernel is the set of elements that goes to 0 under a transformation

Definition

The N-th Dirichlet Kernel is a collection of 2π periodic functions is defined as,

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \quad (4)$$

Dirichlet Kernel

- A Kernel is the set of elements that goes to 0 under a transformation

Definition

The N-th Dirichlet Kernel is a collection of 2π periodic functions is defined as,

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \tag{4}$$

Proof.

$$\begin{aligned} D_N(X) &= e^{-iNx}(1 + \ldots + e^{i2Nx}) \\ &= e^{-iNx}\left(\frac{1-(e^{ix})^{2N+1}}{1-e^{ix}}\right) \\ &= \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}} \times \frac{e^{-\frac{ix}{2}}}{e^{-\frac{ix}{2}}} \end{aligned}$$

Dirichlet Kernel

- A Kernel is the set of elements that goes to 0 under a transformation

Definition

The N-th Dirichlet Kernel is a collection of 2π periodic functions is defined as,

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \tag{4}$$

Proof.

$$\begin{aligned} D_N(X) &= e^{-iNx}(1 + \ldots + e^{i2Nx}) \\ &= e^{-iNx}\left(\frac{1-(e^{ix})^{2N+1}}{1-e^{ix}}\right) \\ &= \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}} \times \frac{e^{-\frac{ix}{2}}}{e^{-\frac{ix}{2}}} \end{aligned}$$

$$\begin{aligned} &= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}} \\ &= \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \end{aligned}$$

□

The Connection

- What was the point of these definitions? It turns out we can do something very nice with how we write our partial sums,

The Connection

- What was the point of these definitions? It turns out we can do something very nice with how we write our partial sums,

A new way to write $S_N(f, x)$

Using Equations (2) and (3) we can write:

$$\begin{aligned} S_N(f,x) &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(k)\, e^{-ink}\, dk \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(k) \left(\sum_{n=-N}^{N} e^{in(x-k)} \right) dk \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(k)\, D_N(x-k)\, dk \end{aligned}$$

The Connection contd.

- This form may seem familiar to you - our partial sum is now in the form of an operation called a convolution

The Connection contd.

- This form may seem familiar to you - our partial sum is now in the form of an operation called a convolution
- For now, we can understand a convolution as an operation on two functions, like multiplication or addition, that gives us a third function

The Connection contd.

- This form may seem familiar to you - our partial sum is now in the form of an operation called a convolution
- For now, we can understand a convolution as an operation on two functions, like multiplication or addition, that gives us a third function

Definition

The convolution of two functions $f(x)$ and $g(x)$ is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Where $*$ is the convolution operator

Definition

We can redefine the partial sums of the Fourier Series as:

The Connection contd.

- This form may seem familiar to you - our partial sum is now in the form of an operation called a convolution
- For now, we can understand a convolution as an operation on two functions, like multiplication or addition, that gives us a third function

Definition

The convolution of two functions $f(x)$ and $g(x)$ is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Where $*$ is the convolution operator

Definition

We can redefine the partial sums of the Fourier Series as:

$$S_N(f,x) = (f * D_N)(x) \tag{5}$$

Cesàro Summation

Definition

The N-th Cesàro sum of a series is defined as the sequence of arithmetic means of the first N partial sums of that series.

Cesàro Summation

Definition

The N-th Cesàro sum of a series is defined as the sequence of arithmetic means of the first N partial sums of that series. Mathematically,

$$\sigma_N = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N(f, x)$$

Cesàro Summation

Definition

The N-th Cesàro sum of a series is defined as the sequence of arithmetic means of the first N partial sums of that series. Mathematically,

$$\sigma_N = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N(f, x)$$

We say the series is Cesàro summable if σ_N converges to $L \in \mathbb{R}$ as $N \to \infty$.

- Why do we talk about Cesàro sums? It turns out it's better to work with arithmetic means of partial sums than partial sums themselves.

Cesàro Summation

Definition

The N-th Cesàro sum of a series is defined as the sequence of arithmetic means of the first N partial sums of that series. Mathematically,

$$\sigma_N = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N(f,x)$$

We say the series is Cesàro summable if σ_N converges to $L \in \mathbb{R}$ as $N \to \infty$.

- Why do we talk about Cesàro sums? It turns out it's better to work with arithmetic means of partial sums than partial sums themselves.
- Our Cesàro limit L is equal to the usual limit if it exists.

Cesàro Summation

Definition

The N-th Cesàro sum of a series is defined as the sequence of arithmetic means of the first N partial sums of that series. Mathematically,

$$\sigma_N = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N(f, x)$$

We say the series is Cesàro summable if σ_N converges to $L \in \mathbb{R}$ as $N \to \infty$.

- Why do we talk about Cesàro sums? It turns out it's better to work with arithmetic means of partial sums than partial sums themselves.
- Our Cesàro limit L is equal to the usual limit if it exists.
- The Cesàro limit may exist even if the usual limit does not.

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1} = \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}$$

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1} = \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}$$

- Since the N-th Fejér Kernel is the arithmetic mean of the first N Dirichlet Kernel...

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1} = \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}$$

- Since the N-th Fejér Kernel is the arithmetic mean of the first N Dirichlet Kernel...
- ...and the N-th Cesàro sum of the Fourier Series is the arithmetic mean of the first N partial sums...

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1} = \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}$$

- Since the N-th Fejér Kernel is the arithmetic mean of the first N Dirichlet Kernel...
- ...and the N-th Cesàro sum of the Fourier Series is the arithmetic mean of the first N partial sums...
- and recalling Equation 5,

Fejér Kernel

Definition

We define the N-th Fejér Kernel as the Cesàro sum of the Dirichlet Kernel,

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1} = \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}$$

- Since the N-th Fejér Kernel is the arithmetic mean of the first N Dirichlet Kernel...
- ...and the N-th Cesàro sum of the Fourier Series is the arithmetic mean of the first N partial sums...
- and recalling Equation 5,

Definition

We can redefine the Cesàro sum of the partial sums of the Fourier Series:

$$\sigma_N(f,x) = (f * K_N)(x) \tag{6}$$

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ *be a family of good kernels and let* f *be an integrable function on the circle.*

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ be a family of good kernels and let f be an integrable function on the circle. Then whenever f is continuous at x,

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ *be a family of good kernels and let* f *be an integrable function on the circle. Then whenever* f *is continuous at* x*,*

$$\lim_{N\to\infty}(f * K_N)(x) = f(x)$$

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ be a family of good kernels and let f be an integrable function on the circle. Then whenever f is continuous at x,

$$\lim_{N\to\infty} (f * K_N)(x) = f(x)$$

If f is continuous, then convergence is uniform on $[-\pi, \pi]$.

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ be a family of good kernels and let f be an integrable function on the circle. Then whenever f is continuous at x,

$$\lim_{N\to\infty}(f * K_N)(x) = f(x)$$

If f is continuous, then convergence is uniform on $[-\pi, \pi]$.

- However, notice that f only being continuous, or f only being integrable does not guarantee convergence.

Convergence and Divergence

Theorem

Let $\{K_N\}_{n=1}^{\infty}$ *be a family of good kernels and let* f *be an integrable function on the circle. Then whenever* f *is continuous at* x*,*

$$\lim_{N\to\infty} (f * K_N)(x) = f(x)$$

If f *is continuous, then convergence is uniform on* $[-\pi, \pi]$*.*

- However, notice that f only being continuous, or f only being integrable does not guarantee convergence. Indeed,

Theorem

There is a function g *which is* 2π *periodic and continuous for which:*

$$\limsup_{N\to\infty} S_N(0) = \infty$$

Where $S_N(0)$ *is the partial sum of the Fourier Series for* g*, evaluated at* $x = 0$*.*

Applications

- Solving PDEs
    - Heat Equation
    - Waves and Vibrations
- Signal Processing
- Acoustics (Noise Removal, Filtering, etc)

Thank you!