An Exposition to Fourier Series

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1 Introduction

1.1 Definition

The aim of a Fourier Series is to express a periodic function in terms of sines and cosines (the purpose of this will be discussed later in this paper). We define a periodic function for which the following holds true:

$$f(x) = f(x + nT)$$

where n is any integer and T is the period of the function. since the Fourier Series is just a sum of sines and cosines, we can define the Fourier Series of a function $f(x)$ as the following:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1}$$

where a_0 is simply the cosine term evaluated at n=0 (this is not needed for sine, since at n=0, the sine term is 0), and a_n and b_n are the coefficients of the cosine and sine terms respectively.

1.2 Orthogonality of Functions

In order to compute the coefficients a_0, a_n and b_0, we have to use the orthogonality of functions. Similar to how two vectors are said to be orthogonal if their dot product is 0, two functions $g(x)$ and $h(x)$ are said to be orthogonal if

$$\int_a^b g(x)h(x) = 0$$

Using orthogonality, we can come to the following conclusions that will help us compute the coefficients:

$$\begin{aligned} &\text{For m, n} \in \mathbb{N} \,\text{and}\, m \neq n \\ &\int_{-\pi}^{\pi} \cos(nx)\,\cos(mx)\,dx = 0 \\ &\int_{-\pi}^{\pi} \sin(nx)\,\sin(mx)\,dx = 0 \\ &\text{For m, n} \in \mathbb{N} \\ &\int_{-\pi}^{\pi} \sin(nx)\,\cos(mx)\,dx = 0 \end{aligned} \tag{2}$$

1.3 Computing a_0, a_n and b_n

We can use the general form of the Fourier Series shown in 1, and the results shown in 2 to compute a_0, a_n and $b_n 0$. Let us first multiply both sides of the equation by $\cos(kx)$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx = a_k \int_{-\pi}^{\pi} (\cos(kx))^2\,dx \tag{3}$$

because all the sines are orthogonal to the $\cos(kx)$ term. All the cosine terms are also orthogonal to the $\cos(kx)$ term except for the singular case where $n = k$. The integral of $(\cos(kx))^2$ for any value of k is simply π. Thus, our equation is simplified to

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx &= a_k \pi \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx \end{aligned} \tag{4}$$

b_k would be calculated in exactly the same way; we would multiply both sides of Equation 1, and would get exactly the same equation as Equation (4), with a sine instead of a cosine - once again, due to orthogonality, all the cosine terms and all but one of the sine terms would drop out, and we would get:

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\sin(kx)\,dx \tag{5}$$

a_0 is just a special case of Equation (3):

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,\cos(kx)\,dx &= a_k \int_{-\pi}^{\pi} (\cos(0))^2\,dx \\ &= a_k \int_{-\pi}^{\pi} 1 \\ &= a_k 2\pi \end{aligned}$$

Thus

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{6}$$

which is just the average of a_0.

2 Complex form

2.1 Representation

Thanks to Euler's formulas, we can rewrite the Fourier Series in complex form:

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\
&= a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{inx}+e^{-inx}}{2} + b_n \frac{e^{inx}-e^{-inx}}{2i}\right) \\
&= a_0 + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}e^{inx} + \sum_{n=1}^{\infty}\frac{a_n + ib_n}{2}e^{-inx} \\
&= a_0 + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}e^{inx} + \sum_{n=-\infty}^{-1}\frac{a_{-n} + ib_{-n}}{2}e^{inx}
\end{aligned}$$

Since the cos term in a_n has n, and the sine term in b_n has n,

$$\begin{aligned}
a_n &= a_{-n} \\
b_n &= -b_{-n}
\end{aligned}$$

Writing the a_0 term as part of our cosine sum, we get:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{a_n - ib_n}{2}e^{inx} + \sum_{n=-\infty}^{-1}\frac{a_n - ib_n}{2}e^{inx} \\
f(x) &= \sum_{-\infty}^{\infty} c_n e^{inx}
\end{aligned}$$

c_n is then

$$\begin{aligned}
c_n &= \frac{a_n - ib_n}{2} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ (\cos(nx) - i\sin(nx))\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx
\end{aligned} \tag{7}$$

2.2 Partial Sums

One of the reasons that the complex representation of the Fourier Series is so useful is because of how effective and natural it is to use when we work with partial sums. We will define the N-th partial sum $S_N(f,x)$ of the Fourier Series as

$$S_N(f,x) = \sum_{k=-N}^{N} c_k e^{ikx} \tag{8}$$

When N is a non-negative integer. Let us define the N-th Dirichlet Kernel, which is a collection of periodic functions defined as:

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} \tag{9}$$

The kernel functions are 2π periodic We can prove that this is equal to $\frac{\sin((n+\frac{1}{2})x}{\sin(\frac{x}{2})}$:

$$\begin{aligned} D_N(x) &= e^{-iNx} + \ldots + e^{iNx} \\ &= e^{-iNx}(1 + \ldots + e^{i2Nx}) \end{aligned}$$

Inside the brackets, we have a geometric series with a common ratio r of e^{ix}. Recall that the sum of a geometric series is $\frac{a(1-r^{n+1})}{1-r}$. Then our expression for $D_N(x)$ becomes:

$$\begin{aligned} D_N(x) &= e^{-iNx}\left(\frac{1-(e^{ix})^{2N+1}}{1-e^{ix}}\right) \\ &= \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}} \\ &= \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}} \times \frac{e^{-\frac{ix}{2}}}{e^{-\frac{ix}{2}}} \\ &= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}} \\ &= \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})} \end{aligned}$$

It turns out we can represent the partial sums of the Fourier Series using the Dirichlet Kernel. Using (7) and (8), we get:

$$\begin{aligned} S_N(f,x) &= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(k)\, e^{-ink}\, dk\right) e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(k)\left(\sum_{n=-N}^{N} e^{in(x-k)}\right) dk \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(k)\, D_N(x-k)\, dk \end{aligned}$$

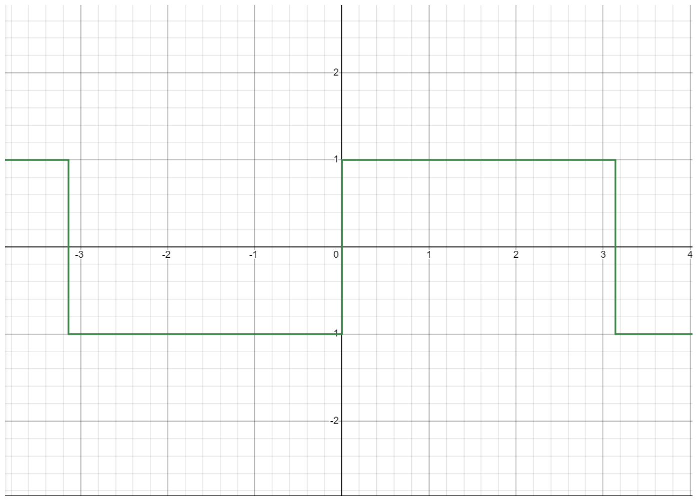

Figure 1: The Square Wave

We have therefore shown that:

$$S_N(f,x) = (f * D_N)(x) \tag{10}$$

Where $*$ is a convolution operator (more in Appendix A).

These partial sums can then help us in approximating the Fourier Series and resolving the Gibbs Phenomenon (see subsection 4.1).

3 An Example of Use of the Fourier Series

Now that we have computed the coefficients, we can use the Fourier Series to approximate various functions. We will compute the Fourier Series for the Square Wave. Since it is periodic, we only need to define it for one period:

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ -1 & -\pi \leq x < 0 \end{cases}$$

Clearly, the function is odd, and so there will be no cosines (since cosines are only even). In other words, a_n will be 0 for all cosine terms. Let us compute the coefficient b_k for the sine terms:

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x)\, \sin(kx)\, dx$$

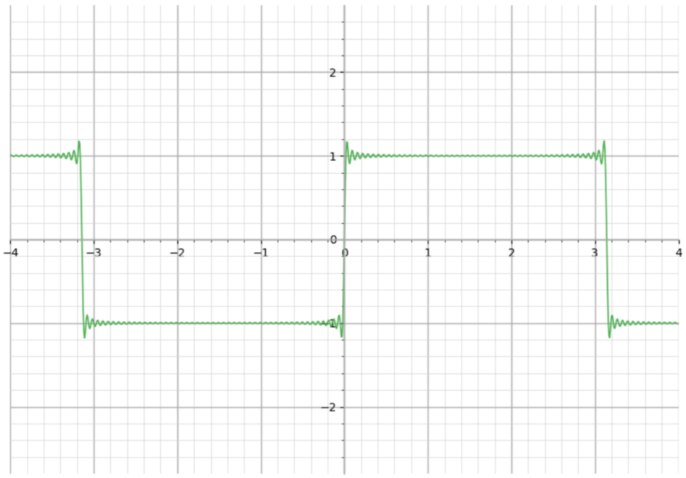

Figure 2: Fourier Series Approximation of the Square Wave - computed till n = 500

We can do this since both $f(x)$ and $\sin(kx)$ are odd, and so their product is even. Notice that from 0 to π, $f(x)$ is simply 1. Thus:

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(kx)\, dx \\ &= \frac{2}{\pi} \cdot \left.\frac{-\cos(kx)}{k}\right|_0^{\pi} \end{aligned}$$

Depending on whether k is even or odd, we get different solutions. For even values of k, the integral evaluates to 0:

$$\frac{2}{\pi} \cdot \left.\frac{-\cos(kx)}{k}\right|_0^{\pi} = \begin{cases} \frac{4}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Since all sine terms with even values of b_n are killed, we get the following Fourier Series for the Square Wave:

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin(nx)$$

As can be seen in Figure 2, even computed to 500 terms, the Fourier Series is a very good approximation of the Square Wave.

4 Cesàro Summation, Gibbs Phenomenon

4.1 The Gibbs Phenomenon

We notice a problem in Figure 2; at the edges of the jumps in the Square Wave, the Fourier Series overshoots and undershoots. At these discontinuities, it is a poor approximation of the function. This is not surprising; it does not seem possible to reconstruct a discontinuous function from a sum of continuous ones. We can say we can reconstruct the discontinuous function 'almost everywhere', except at the points of discontinuity. This 'rippling' around the discontinuities is called the Gibbs phenomenon. It does not shorten as the number of terms in our sum goes to infinity, but it does narrow.

The Fourier Series - or more precisely, its finite sums - are **not** a replica of our original function. To resolve this, and the Gibbs Phenomenon, we will use the Cesàro summation of the partial sums of the Fourier Series.

4.2 Cesàro Summation

Let $f_n(x)$ be a series, and

$$S_N(f,x) = \sum_{n=0}^{N} f_n(x)$$

be the N-th partial sum of that series. Then the N-th Cesàro sum σ_N is:

$$\sigma_N = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} S_N(f,x) \tag{11}$$

We say the series is Cesàro summable if σ_N converges to $L \in \mathbb{R}$ as $N \to \infty$.

What is the point of these Cesàro sums? It turns out these averages show better behaviour than partial sums in the sense that our Cesàro limit (if it exists) will equal the usual limit (when it exists) and may even exist when the usual limit does not.

4.3 The Fejér Kernel

The N-th Fejér Kernel is simply the Cesàro sum of the partial sums of the Dirchilet Kernels. We prefer it over the Dirchilet Kernel because it is a good kernel, whereas the latter is a bad one (see B for more). From the way we defined the Cesàro sum, we can see the Fejér Kernel is a sort of averaging of the Dirchilet Kernels - it makes sense, then, that we might be able to use it to 'smooth' the overshoot and undershoot that is the Gibbs Phenomenon. Let us be a bit more mathematically rigorous; the Fejér Kernel is defined as:

$$K_N(x) = \frac{\sum_{n=0}^{N} D_N(x)}{N+1}$$

From Equation (9),

$$\begin{aligned}
K_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} e^{ikx} \\
&= \frac{1}{N+1}\left(\sum_{k=0}^{0} e^{ikx} + \sum_{k=-1}^{1} e^{ikx} + \sum_{k=-2}^{2} + \ldots + \sum_{k=-n}^{n} e^{ikx}\right) \\
&= \frac{1}{N+1}\sum_{k=-N}^{N}(n+1-|k|)e^{ikx}
\end{aligned}$$

Then the Cesàro sum of the partial sums of the Fourier Series is given as follows:

$$\sigma_N(f,x) = (f * K_N)(x) \tag{12}$$

This can be seen by the fact that the N-th Cesàro sum is the arithmetic mean of the partial sums, and so must be the convolution of f with the arithmetic mean of the Dirichlet Kernel (keeping in mind Equation (9)). Before proceeding, let us derive the trigonometric representation of the Fejér Kernel:

$$\begin{aligned}
K_N(x) &= \frac{\sum_{n=0}^{N} D_N(x)}{N+1} \\
&= \frac{\sum_{n=0}^{N} \sin((n+\frac{1}{2})x)}{(N+1)(\sin(\frac{x}{2}))} \\
&= \frac{\sum_{n=0}^{N} \sin((n+\frac{1}{2})x)}{(N+1)(\sin(\frac{x}{2}))} \times \frac{\sin(\frac{x}{2})}{\sin(\frac{x}{2})} \\
&= \frac{\sum_{n=0}^{N} \sin(\frac{x}{2})\sin((n+\frac{1}{2})x)}{2(N+1)(\sin^2(\frac{x}{2}))} \\
&= \frac{\sum_{n=0}^{N} \cos(nx) - \cos((n+1)x)}{2(N+1)(\sin^2(\frac{x}{2}))}
\end{aligned}$$

Which is a telescoping sum - all but 2 terms cancel. We thus get:

$$\begin{aligned}
K_N(x) &= \frac{1-\cos((N+1)x)}{2(N+1)(\sin^2(\frac{x}{2}))} \\
&= \frac{\sin^2(\frac{(N+1)x}{2})}{(N+1)(\sin^2(\frac{x}{2}))}
\end{aligned}$$

as the trigonometric representation of the Fejér Kernel. It is immediately obvious from this trigonometric representation that the Fejér kernel is **always**

non-negative - this is an incredibly important result to resolve the Gibbs phenomenon.

How is the Fejér kernel relevant? Equation (12) shows us that the Cesàro sum of the partial sums - its average, so to speak - is simply a convolution of $f(x)$ and $K_N(x)$. Essentially, when we replace our partial sums with their Cesàro sums, we smooth the overshoot and undershoot that is the Gibbs phenomenon, by smoothing and averaging it, so to speak.

5 Convergence

Let us put Cesàro sums on one side for a moment, and review the fundamental claim made by Fourier as in Equation (1). Generally, we can show pointwise convergence under the following conditions:

Theorem 1 *Let f be a 2π periodic function that is continuous and has a bounded continuous derivative, except, possible at a finite number of points. Then Equation* (1) *holds at every $x \in \mathbb{R}$ where f is continuous.*

Though we have discussed how the Fejér kernel and Cesàro sums of the partial sums of the Fourier Series are vital in mitigating the Gibbs Phenomenon, we have not yet discussed on of the key benefits of these techniques, which stems from the fact that the Fejér kernel is a good kernel and the Dirichlet Kernel is a bad kernel (see Appendix B for more). There are two significant results have to do with the convergence of the Fourier Series

Theorem 2 *Let $\{K_N\}_{n=1}^{\infty}$ be a family of good kernels and let f be an integrable function on the circle. Then whenever f is continuous at x,*

$$\lim_{N\to\infty} (f * K_N)(x) = f(x)$$

If f is continuous, then convergence is uniform on $[-\pi, \pi]$.

Let us prove this: f $\epsilon > 0$ and f is continuous at x, choose δ such that if $|y| < \delta$, then $|f(x-y) - f(x)| < \epsilon$.

Using the first property of good kernels, we can write

$$\begin{aligned}(f * K_N)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y) f(x-y)\,dy - f(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)[f(x-y) - f(x)]\,dy.\end{aligned}$$

Taking the absolute value,

$$\begin{aligned}
|(f * K_N)(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(y)[f(x-y) - f(x)]\, dy \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_N(y)||f(x-y) - f(x)|\, dy \\
&\leq \frac{1}{2\pi} \int_{|y|<\delta} |K_N(y)||f(x-y) - f(x)|\, dy \\
&+ \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_N(y)||f(x-y) - f(x)|\, dy \\
&\leq \frac{1}{2\pi} \left(\epsilon \int_{|y|<\delta} |K_N(y)|\, dy + 2B \int_{\delta \leq |y| \leq \pi} |K_N(y)|\, dy \right),
\end{aligned}$$

where B is a bound for $|f|$.

The second property of good kernels shows that

$$\frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_N(y)|\, dy \leq \frac{\epsilon M}{2\pi},$$

for all $n \geq 1$. The third property of good kernels shows that

$$\frac{2B}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_N(y)|\, dy \leq \epsilon,$$

for all $n \geq N(\delta)$. Therefore, we have

$$|(f * K_N)(x) - f(x)| \leq C\epsilon.$$

With $C\epsilon$ becoming arbitrarily small as ϵ goes to 0.

6 Divergence

Note that Thoerem 2 does not guarantee that a continuous function will converge. Indeed, it was proved by Du-Bois Reymond that:

Theorem 3 *There is a function g which is 2π periodic and continuous for which:*

$$\limsup_{N \to \infty} S_N(0) = \infty$$

Where $S_N(0)$ is the partial sum of the Fourier Series for g, evaluated at $x = 0$.

Similarly, for integrable functions, Kolmogorrof proved:

Theorem 4

7 Using the Fourier Series to Solve PDEs

One of the most useful ways to employ the Fourier Series is while solving Partial Differential Equations.

7.1 Laplace Equation

The Laplace Equation is one of the most fundamental second-order PDEs and arises in the heat and diffusion equations. In the study of heat - which we will use as an example - it is the steady state heat equation. The differential equation is as follows:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

We will use the Fourier Series to solve the Laplace equation inside a circle. Our heat source is a point source on the boundary of a unit circle on the x axis, and on the rest of the body, the temperature is 0. In other words, the boundary function is, for all practical purposes, a Dirac Delta function. Since we are dealing with circles, we will use polar coordinates, and we will define U as

$$U(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} a_n r^n \sin(n\theta)$$

which satisfies the Laplace equation. We must now use boundary conditions to solve it. As mentioned above, on the boundary, we have the Dirac Delta function:

$$\begin{aligned} \delta(\theta) &= U(1, \theta) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) \end{aligned}$$

We have no sines, as the Delta function is even. Let us now compute the coefficients:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{\pi}^{\pi} \delta(\theta)\, d\theta \\ &= \frac{1}{2\pi} \end{aligned} \tag{13}$$

where a_0 is the average value of the temperature.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-pi}^{\pi} \delta(\theta)\, \cos(n\theta)\, d\theta \\ &= \frac{1}{\pi} \end{aligned}$$

Substituting the coefficients back into our equation for U:

$$U(r,\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \cos(n\theta)$$

A Appendix: Convolution

Just like addition and multiplication, a convolution is an operation we perform on two functions that gives us a third function. Mathematically, we define the convolution of two functions $f(x)$ and $g(x)$ as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Where t is some constant. We are essentially shifting $g(x)$ over $f(x)$, with the integral giving us the overlap.

Some of the properties of the convolution operation are given ahead. It is commutative; that is,

$$x * h = h * x$$

It is associative; that is,

$$x * (h_1 * h_2) = (x * h_1) * h_2$$

It is distributive; that is,

$$x * (h_1 + h_2) = x * h_1 + x * h_2$$

B Appendix: Good and Bad Kernels

A family of integrable functions $\{K_N\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if it satisfies the following 3 properties:

1.

$$\text{For all } n \geq 1, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\, dx = 1 \tag{14}$$

2.

$$\text{There exists } M > 0 \text{ such that for all } n \geq 1, \quad \int_{-\pi}^{\pi} |K_N(x)|\, dx \leq M \tag{15}$$

3.

For every $\delta > 0$

$$\int_{\delta \leq |x| \leq \pi} |K_N(x)|\, dx \to 0 \tag{16}$$

as $n \to \infty$