(Kirchoff's) Matrix-Tree Theorem

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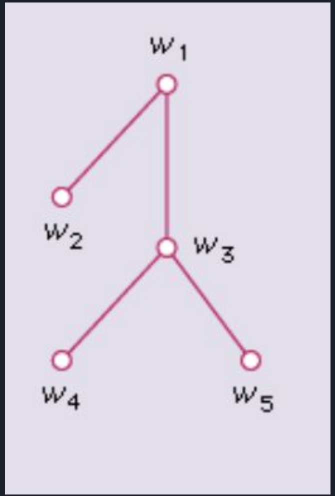
What is the matrix-tree theorem?

- The matrix-tree theorem allows us to count the number of spanning trees in a graph
- Extremely useful in many fields that make use of graphs (computer science, quantum physics, circuitry, … etc.)



History of the Matrix-Tree Theorem

- Gustav Kirchhoff (1847)
  - Introduced Kirchhoff's circuit laws describing electric current flow.
  - Formulated an early method to count spanning trees using determinants.

- Arthur Cayley (1889)
  - Derived Cayley's formula for counting labeled trees.
  - Influenced the study of spanning trees.

- 20th Century Developments
  - Formal proof and broader advancements in algebraic graph theory.

The Theorem Statement

- $\tau(G) = \det(L_{ii})$
- Where $\tau(G)$ represents the total number of spanning trees in graph G, and L_{ii} represents any cofactor of the **Laplacian matrix** for graph G

Definitions and Notation

Graph G = (V, E)

- Vertex set V = {v1, v2, v3, v4, … etc.}
- Edge set includes all edges/connections between 2 vertices
- Spanning tree: subgraph that includes all vertices of G and is a tree (connected and acyclic)

Degree of a Vertex v_i

- Denoted by $deg(v_i)$
- Number of edges incident to v_i

Definitions and Notation (Continued)

Adjacency Matrix A

- n×n matrix
- $A_{ij} = 1$ if there is an edge between v_i and v_j
- $A_{ij} = 0$ otherwise

Degree Matrix D

- Diagonal matrix
- $D_{ii}=\deg(v_i)$

Laplacian Matrix L

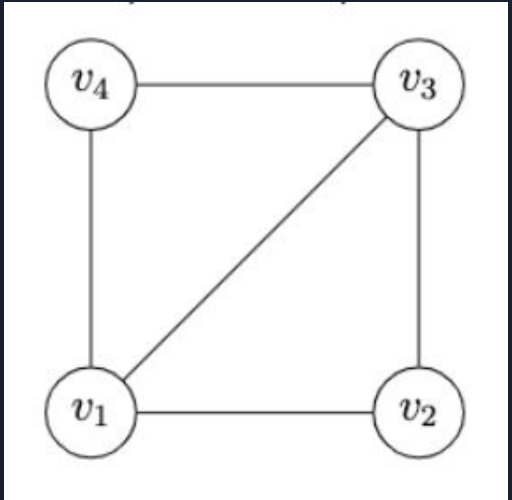
- Defined as: $L = D - A$

Definitions and Notations (Continued)

Signed Incidence Matrix B

- $n \times m$ matrix
- B_{ij} = 1 if edge *j* is incident to vertex *i*
- −1 if it is incident to vertex *i* in the opposite direction
- 0 otherwise

Example



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{L}_{11} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(1) $(v_1, v_2), (v_2, v_3), (v_3, v_4)$
(2) $(v_1, v_2), (v_2, v_3), (v_1, v_4)$
(3) $(v_1, v_2), (v_1, v_3), (v_3, v_4)$
(4) $(v_1, v_2), (v_1, v_4), (v_3, v_4)$
(5) $(v_2, v_3), (v_3, v_4), (v_1, v_4)$
(6) $(v_2, v_3), (v_1, v_3), (v_1, v_4)$
(7) $(v_1, v_2), (v_1, v_3), (v_4, v_3)$
(8) $(v_1, v_3), (v_2, v_3), (v_1, v_4)$

There are indeed 8 spanning trees in G.

Proofs of the Matrix-Tree Theorem

- Traditional Proof, making use of the Cauchy-Binet Theorem
- Random Walk Proof, making use of Markov Chain concepts
- Probabilistic Proof, making use of generating functions
- Topological Proof, making use of Homology Groups

Big Application: Cayley's Theorem

- The matrix-tree theorem can be used to prove Cayley's Theorem, which states that the number of distinct labeled trees on n vertices is n^{n-2}

- Wide-ranging applications
  - Designing robust communications networks
  - Ensuring reliable power supply from electrical grids

- Graph Algorithms in Computer Science

- Understanding the robustness of market structures

The Tutte Polynomial and other Specializations

- Encodes combinatorial properties of a graph

1. **Initial Condition**: If G has no edges, then $T(G; x, y) = 1$.
2. **Deletion-Contraction Recurrence**: If e is an edge of G,

$$T(G; x, y) = \begin{cases} T(G - e; x, y) + T(G/e; x, y), & \text{if } e \text{ is neither a loop nor a bridge,} \\ x \cdot T(G/e; x, y), & \text{if } e \text{ is a bridge,} \\ y \cdot T(G - e; x, y), & \text{if } e \text{ is a loop.} \end{cases}$$

- Key Definitions:
  - Loop: An edge that connects a vertex to itself
  - Bridge: An edge whose removal increases the number of connected components in a graph
  - Deletion: Removing an edge from the graph
  - Contraction: merging the two vertices of an edge into a single vertex and removing
  - any loops or duplicate edges

Specializations of the Tutte Polynomial

Chromatic Polynomial: counts the number of ways to color the vertices of G using k colors such that no two adjacent vertices share the same color

Flow Polynomial: counts the number of nowhere-zero k-flows in G (A k-flow assigns a flow to each edge such that the flow is conserved at each vertex, and no flow is zero on any edge)

Reliability Polynomial: gives the probability that a network remains connected when edges fail independently with probability p

$$\tau(G) = T(G; 1, 1).$$

Applications - Network Reliability Analysis

- Reliability Polynomial *R(G, p)*
  - Expresses network reliability, where p is the probability that an edge functions correctly
  - Provides the probability that a network remains connected
  - Coefficients often involve the count of spanning trees, as given by the Matrix-Tree Theorem
- The more spanning trees a network has, the most robust it generally is

Applications - Electrical Circuits

- Each spanning tree of a graph representing an electrical network corresponds to a unique way of maintaining current flow through the network without forming cycles

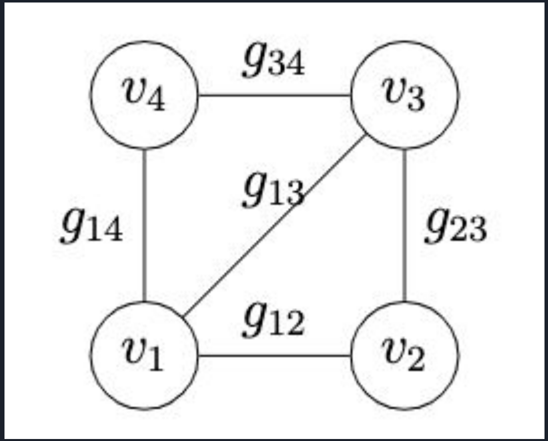
- Closely tied to:
  - Kirchoff’s Laws
  - Effective Resistance

Electrical Circuits - Kirchoff’s Laws

- Current Law: Total current entering a node equals the total current leaving the node
- Voltage Law: Sum of electrical potential differences around any closed loop must be zero
- The Laplacian Matrix represents these laws with matrix elements corresponding to conductances (inverse of resistance) of network edges
- Counting the number of spanning trees with the Matrix-Tree Theorem helps solve for currents and voltages in a network, according to Kirchoff’s Laws

Effective Resistance

- The effective resistance R_{ij} between nodes i and j in a network can be derived from the total number of spanning trees that include the edge (i, j)



$$\mathbf{L} = \begin{pmatrix} g_{12}+g_{13}+g_{14} & -g_{12} & -g_{13} & -g_{14} \\ -g_{12} & g_{12}+g_{23} & -g_{23} & 0 \\ -g_{13} & -g_{23} & g_{13}+g_{23}+g_{34} & -g_{34} \\ -g_{14} & 0 & -g_{34} & g_{14}+g_{34} \end{pmatrix}$$

We can use the Matrix-Tree Theorem to give us the number of spanning trees in the network, reflecting the different ways the network can maintain connectivity and current flow, despite failures

Applications in Quantum Graphs

- Quantum Graphs
  - Mathematical models representing quantum systems
    - Edges: One-dimensional quantum wires
    - Vertices: Scattering centers
  - Matrix-Tree Theorem aids in studying spectral properties

- Spectral Properties
  - Eigenvalues of the Laplacian matrix L provide information about energy level
  - Matrix-Tree Theorem helps count spanning trees
  - Analyzing spanning trees influences the spectral properties of the quantum graph

Quantum Transport

- Analyzed in mesoscopic systems using the Matrix-Tree Theorem
- Electrons travel through a network of quantum wires
- Transport properties influenced by network's connectivity

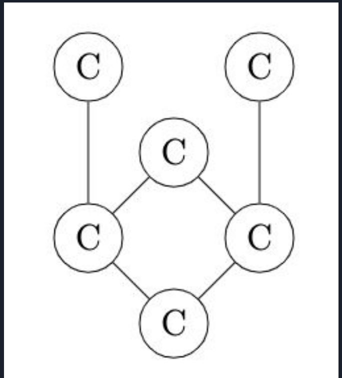
- Role of Spanning Trees
  - Critical in determining paths available for electron transport
  - Each spanning tree: Unique configuration of quantum paths

Molecular Structures and Chemical Graph Theory

- Molecular Structures in Quantum Mechanics
  - Atoms represented as vertices and bonds as edges in a molecular graph
  - Matrix-Tree Theorem helps analyze stability and reactivity by counting spanning trees
- Chemical Graph Theory
  - Uses the Matrix-Tree Theorem to predict properties of chemical compounds
  - Number of spanning trees correlates with the stability of the molecule
  - Higher number of spanning trees indicates greater stability

Vibrational Modes of Molecules

- Influenced by the structural properties of the molecule
- Matrix-Tree Theorem provides insights into the graph’s connectivity
- Eigenvalues of the Laplacian matrix relate to vibrational frequencies



Benzene Molecule, represented as a graph

Quantum Entanglement

Entanglement Measures:

- Degree of entanglement related to graph connectivity
- Matrix-Tree Theorem quantifies connectivity by counting spanning trees, provides a measure of entanglement

Entanglement Robustness:

- Robustness analyzed by studying spanning trees
- Higher number of spanning trees indicates more robust entanglement
- Independent paths help maintain entanglement

Thank You for Listening!