# COMBINATORICS ON WORDS

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## Abstract

Words appear in our everyday life. I began this research on words because math seems closely related to them in some ways as sequence of numbers can also be looked at as a word, and I wanted to find out applications of math into languages. In the beginning of the paper, I will introduce the main topics that this paper will cover and give basic mathematical definitions related to words. After that, descriptions and proofs of more interesting topics(Fine and Wilf's Theorem, Sturmian word, Plactic monoid) will be introduced. I hope you find the application of math in the language fascinating. Enjoy!

## Contents



## 1. Introduction

This paper includes descriptions and proofs of words with unique properties: Fine and Wilf's Theorem, Sturmian word, and Plactic monoid. The motivation behind selecting those three topics as main topics of this paper is that Sturmian word seemed to be related to Fibonacci sequence, a topic that I really enjoy learning about, and Plactic monoid which seems to have links to Young tableau. Because I wanted to learn more about fundamental theorems of combinatorics(namely Schensted's algorithm in RSK), I included Plactic monoid as my third topic. This section will include main theorems that we will prove in this paper.

**Theorem 1.1.** *For a word $w$ with two periods $p$ and $q$, if $|w| \geq p + q - gcd(p, q)$, then $w$ has the period $gcd(p, q)$.*

For this theorem, also known as the classic Fine and Wilf's theorem, the proof will involve a stronger version of this theorem with the theorem above being a specific case of the stronger version. Below is the statement of the stronger case of the classic Wilf's theorem.

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**Theorem 1.2.** *Let $w$ be a word with periods $p_1, p_2, p_3$ satisfying $p_1 \leq p_2 \leq p_3$. If $|w| \geq f(p_1, p_2, p_3)$, $w$ also has the period $gcd(p_1, p_2, p_3)$.*

Here, we have not yet defined what $f(p_1, p_2, p_3)$ does, but the main idea is to plug in 0 for $p_1$ to prove the classic version. The rigorous proof will be in the section Fine and Wilf's Theorem.

**Definition 1.3.** A word $w$ is called a Sturmian word if $|w|$ is infinite, is composed of only two distinct alphabets, and contains $n+1$ factors for each $n \geq 0$.

The most notable example of a Sturmian word as we noted earlier is the Fibonacci word. The Fibonacci word is formed by infinitely repeating the concatenation of two previous words in the sequence. The sequence is defined as $F_0 = 1, F_1 = 0, F_n = F_{n-1}F_{n-2}$. So, $F_2 = 01, F_3 = 010$ as an example. So, Fibonacci is in the form $0100101001001\ldots$. If you carefully look at this word, For every factor of length n, there are indeed n+1 ones of them. For example, there are only 3 factors of length 2(01,10,00). Similarly, the condition of Sturmian word is satisfied for all other n. In the section Sturmian Word, we will prove other equivalent definitions of Sturmian Words.

**Definition 1.4.** The plactic monoid on the alphabet A is the quotient $Pl(A) = A^*/\equiv$, where $\equiv$ is the congruence generated by the Knuth relations $xzy \equiv zxy (x \leq y \leq z)$, $yxz \equiv yzx (x < y \leq z)$

Plactic monoid represents an equivalence relationship among different words based on their Schensted tableau. The section about plactic monoid will be the last of three main sections in this paper, and basics of Young tableaux are explained in this section to introduce the basic ideas of plactic monoid.

## 2. Preliminaries

In this section, important definitions regarding words are outlined such that later sections with higher complexity can be understood more easily.

**Definition 2.1.** An operation $*$ is associative if $x*(y*z) = (x*y)*z$ for x,y,z $\in$ an arbitrary set X.

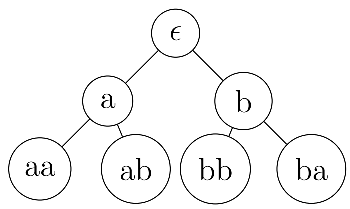
Prime examples of associative operations are addition, multiplication, and division of real numbers. Associativity is an important property that is found in the operation of concatenating two words.

**Definition 2.2.** Semigroup is a set under associative binary operation. Subsemigroup is a subset closed under the binary operation.

For example, the set of words formed from an alphabet is a semigroup. Semigroup is a significant term that describes words in an accurate manner because all words can be formed from applying multiple binary operations on the set of letters that compose the word. A simple example would be forming "math" from letters m,a,t, and h. "math" can be obtained from applying the binary operation between m and a to form "ma". Then, the same operation is applied between "ma" and t to create "mat". Lastly, "mat" and h are combined to form "math". All the words that are formed from applying the binary operation to m,a,t, and h are all elements of a semigroup since they are formed from the binary operation. Even though the following example of matrices as a subsemigroup is a bit far off from words, it still helps us to grasp the definition. The matrix in the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ where $x \in \mathbb{R}$ is a subsemigroup of the group of 2 by 2 matrices under the standard matrix multiplication. The reason for this is that $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} * \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix}$. $x^2$ is also a real number, so we can see that the set of matrices in the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ is indeed closed under the associative binary operation of matrix multiplication.

**Definition 2.3.** A set M is called monoid if M contains a neutral element $\epsilon$ such that m$\epsilon$=$\epsilon$m=m for m $\in$ M. Submonoid is a subset of M such that it is closed under the operation and contains $\epsilon$.

Monoid has a similar feel to a field because a monoid must have the identity element under a binary operation. An example of monoid is the set of all nonnegative integers defined under the operation $min\{x + y, 10\}, x \neq y$. The set $\{0, 3, 4, 7, 8, 9\}$ is a submonoid of the monoid $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Another important visual for a word is the tree diagram that contains all unique words formed from alphabets. Below is an example of a tree diagram representing all the words over two alphabets: a,b.




This tree diagram demonstrates how one can browse a word consisting of a given number of alphabets. For example, the word "ab" is traced from the empty word $\epsilon$ to a, then to "ab". Any word can be obtained this way in a tree diagram which contains all the alphabets that make up the word.

**Definition 2.4.** Let X, Y be two subsets of a semigroup S, direct product of X and Y is the set XY such that xy $\in$ XY for x $\in$ X and y $\in$ Y.

Instead of direct product in other contexts of math, which forms a tuple, a direct product in this paper refers to concatenating two different elements into one. To give a specific example, let $X = \{nikash, ryan, woong\}$ and let $Y = \{isfast, isanorca, isagorilla$. $XY = \{nikashisfast, nikashisanorca, nikashisagorilla, ryanisfast, ryanisanorca, ryanisagorilla$ $woongisfast, woongisanorca, woongisagorilla\}$

**Definition 2.5.** The set $X^+$ is a semigroup produced from X defined by $X^+ = \{x_1 \cdots x_n \,|\, n \geq 1, x_i \in X\}$ from X.

We can observe that $X^+$ is a semigroup because the set is constructed from combining two or more elements in X according to the definition of $X^+$. The formation of the words from the alphabets "m","a","t","h" is considered $X^+$ based on defining $X = \{m, a, t, h\}$. So, in total, there will be a total of $4 + 12 + 24 + 24 = 64$ elements in $X^+$ excluding the neatural eleemtn $\epsilon$. Here, we counted the total number of elements of length 1, 2, 3, and 4 while considering different ways to order those elements. We excluded the listing of all elements because it would be redundant.

**Definition 2.6.** The operation of generating the submonoid $X^*$ from X is called the *star operation*. $X^* = X^+ \bigcup \{\epsilon\}$.

**Definition 2.7.** A word $w$ is called a factor of $u$ if $\exists$ words a,b such that $u = awb$. $w$ is a $proper\, factor$ if $ab \neq \epsilon$. Furthermore, the set of all factors of $w$ is denoted as $F(x)$.

For example, the word "is" is a proper factor of "nikashisfast". And, the word "nikashisfast" is a factor of "nikashisfast". The definition of proper factor is essentially same as the definition of proper divisor. Both definitions of an object consider objects satisfying a condition with the exception of itself.

**Definition 2.8.** Let $w = a_1a_2 \cdots a_n$ where $a_i$ is an alphabet. The reversal of w is denoted as $\tilde{w}$ which equals to $a_na_{n-1} \cdots a_1$.

The definition of reversal leads to another word with a unique property called palindrome.

**Definition 2.9.** A word $w$ is a *palindrome word* if $w = \tilde{w}$.

Perhaps, palindrome is one of the most commonly known type of word with a unique property. Mathematically, palindromes are even more intriguing to study because of the ways other words are defined using palindromes. Below, we prove more specific conditions of for a word to be a palindrome.

**Theorem 2.10.** *In the case of $|w| \equiv 1 (mod\, 2)$, $w$ is a palindrome if and only if $w = xa\tilde{x}$ for some letter $a$ and word $x$ with $|x| \geq 0$. In the case of $|w| \equiv 0(mod\, 2)$, $w$ is a palindrome if and only if $w = x\tilde{x}$ for word $x$ with $|x| \geq 0$.*

*Proof.* For the case $|w| \equiv 1(mod\, 2)$, If $w = xa\tilde{x}$, $\tilde{w} = \tilde{\tilde{x}}a\tilde{x} = xa\tilde{x}$ since $\tilde{\tilde{x}} = x$. Since $w = \tilde{w}$, $w$ is a palindrome. If w is a palindrome, where $w = a_1a_2 \cdots a_n$, Since $w = \tilde{w}$ by definition, $w = a_1a_2 \cdots a_{(n-1)/2}ba_{(n-1)/2} \cdots a_2a_1$, where b is a letter in at the middle index of $w$. The factor of $w$, $a_1 \cdots a(n-1)/2$ can be seen as word $x$, then $w$ is in the form $xb\tilde{x}$, which is equivalent to $w$ in the form of $xa\tilde{x}$. For the case $|w| \equiv 0(mod\, 2)$, If $w = x\tilde{x}$, $\tilde{w} = \tilde{\tilde{x}}\tilde{x} = x\tilde{x}$. Since $w = \tilde{w}$, $w$ is a palindrome. If $w$ is a palindrome, meaning $w = \tilde{w}$, let $w = a_1a_2 \cdots a_n$. Because $w = \tilde{w}$, w must be able to be written as $a_1a_2 \cdots a_{n/2}a_{n/2} \cdots a_2a_1$. Let $a_1a_2 \cdots a_{n/2}$ be equal to $x$, then $w = x\tilde{x}$. ∎

Note that for the first part of the proof, the first case is the proof going in the direction from $|w| \equiv 1$ (mod 2) to w being a palindrome, and the second case is the proof going in the other direction. The second part of the proof works in the same manner as before, but considers $|w| \equiv 0$ (mod 2). To make the proof of theorem 2.10 clearer, We will prove the proposition below.

**Proposition 2.11.** *Let word $w$ be represented in the form $w_1w_2 \ldots w_n$ where $w_i$ is a word. $\tilde{w} = \tilde{w_n}\tilde{w}_{n-1} \ldots \tilde{w_1}$.*

*Proof.* Let $n = 1$, $w = \tilde{w_1}$ by the definition of reversal notation. Let $n = k$, assume $\tilde{w} = \tilde{w}_k\tilde{w}_{k-1} \ldots \tilde{w}_1$. For the case $n = k + 1$, let $w = a_1a_2 \ldots a_n = w_1w_2 \ldots w_{k+1}$. Since $\tilde{w} = a_na_{n-1} \ldots a_1$ we know that $\tilde{w}$ should have $\tilde{w}_{k+1}$ as its leftmost factor, representing $a_n \ldots a_{n-|w_{k+1}|+1}$ in $w$. Since we know that we can reverse $w_1w_2 \ldots w_k$ from the case $n = k$, the reversal of the word in $k + 1$ distinct words can be done by adding $\tilde{w}_{k+1}$ as the leftmost factor of $\tilde{w}$ in the previous case. From this, we can observe that a word expressed in $k + 1$ different words also works in the same way as the reversal of $w$ in the case $n = k$. By proof of induction, we have proven that the reversal of a word represented in k distinct words works in the mechanism proposed in the proposition. ■

Note that this proposition involves a word being defined in terms of multiple words, not multiple alphabets. An interesting remark about proposition 2.11 is that same mechanism is used to find the transpose of product of n matrices or the inverse of product of n matrices.

**Definition 2.12.** Let A be the set called alphabet, which is composed of all the letters that make up a set of words. For word $w = a_1a_2 \cdots a_n$, an integer p $\geq$ 1 is called a *period* if $a_i = a_{i+p}$. The smallest p such that this equality holds is called the period.

Observe that at some point, the term $a_{i+p}$ does not exist in the word because $i + p$ might exceed $n - 1$. In this case, p is still considered the period if the equality mentioned in definition 2.12 holds true for smaller indexes.

**Definition 2.13.** A word $w \in A^+$ is *primitive* if $w = u^1$ for some u $\in A^+$

For example, in the tree diagram displayed under definition 2.3, the element "aa" is not primitive because $aa = a^2$. However, ab is primitive.

**Definition 2.14.** Two words x,y are conjugates of each other if $\exists$ words u,v such that $x = uv, y = vu$

The relationship of conjugates can also be perceived as two words being mere periodic shifts of one another. To give a specific example, the word "nikashiscool" and "coolnikashis" are conjugates of one another.

**Corollary 2.15.** *The conjugacy class of a word of length n and period p has p elements if $p \mid n$, and has n elements otherwise*

We will not formally show how this statement is true, but we can imagine shifting first $i$ elements of a word to the back to produce a different word in the conjugacy class. If p $\nmid$ n, then this shifting mechanism will produce a different word for all i $\leq$ n. However, if p $\mid$ n, then shifting the word by more than p would not make a new word in the conjugacy class.

**Definition 2.16.** If x is a prefix of y, x is a substring of y taken from index 0 to i with $i \leq |y| - 1$. Prefix of length j of y is denoted as $\text{pref}_j(y)$.

Any prefix of another word is less in lexicographic order by definition 2.19. A prefix of another word would also be a radix order.

**Definition 2.17.** If x is a suffix of y, x is a substring of y taken from index i to $|y| - 1$ with $i \geq 0$. Suffix of length j of y is denoted as $\text{suff}_j(y)$.

Contrary to prefix, nothing much can be said about suffix of a word in terms of radix order and lexicographic order.

**Definition 2.18.** x is a radix order to y if $|x| \leq |y|$ and $a \leq b$ where $x = uax', y = uby'$ where a,b are letters.

Radix order is a specific case of lexicographic order as you can see the definition of lexicographic order below.

**Definition 2.19.** In lexicographic order, x < y if $x = uax', y = uay'$ where a < b, and x,y are words while a,b are letters.

## 3. Fine and Wilf's Theorem

Fine and Wilf's theorem is one of many theorems that exhibit interesting properties of a word regarding the periods. In this section, we will state and prove the original statement of Fine and Wilf's theorem by proving a more general case.

**Theorem 3.1.** *For a word $w$ with two periods $p$ and $q$, if $|w| \geq p+q-gcd(p,q)$, then $w$ has the period $gcd(p,q)$.*

To prove theorem 3.1, we will prove a stronger case of the theorem, which involves a word with three nonnegative periods. Before proving a stronger version of theorem 3.1, let's define a few important functions.

**Definition 3.2.** Given three integers $p_1, p_2, p_3$ with $p_1 \leq p_2 \leq p_3$, $R(p) = (p_1, p_2 - p_1, p_3 - p_1)$ if $p_1 = 0$, and $R(p) = (0, p_2, p_3 - p_2)$ if $p_1 = 0$.

This function defined as $R(p)$ is also known as Euclidean algorithm, which is frequently used to compute greatest common denominator of n numbers. One example of this function can be done on the triplet $p = (18, 48, 50)$. We will later show how $gcd(p)$ can be computed easily after defining a few more definitions.

**Definition 3.3.** $O(p)$ gives corresponding triplet in the correct order from smallest to biggest.

For example, $O(5,4,3) = (3,4,5)$.

**Definition 3.4.** The function $S(p)$ gives the triplet $S(p) = O(R(p))$.

**Definition 3.5.** Given a triplet p, $|p| = p_1 + p_2 + p_3$

**Definition 3.6.** The recursive function $p^{k+1} = S(p^k)$ for $k \geq 0$. $p^k = (p_1^k, p_2^k, p_3^k).p^0 = p$.

Back to the example of finding $gcd(18, 48, 50)$, definition 3.4 and definition 3.6 really help computing the value efficiently. Repeatedly applying $S(p)$, we get $p^1 = O(18,30,32) = (18,30,32)$ $p^2 = O(18,12,14) = (12,14,18)$ $p^3 = O(12,2,6) = (2,6,12)$ $p^4 = O(2,4,10) = (2,4,10)$ $p^5 = O(2,2,8) = (2,2,8)$ $p^6 = O(2,0,6) = (0,2,6)$ $\cdots$ $p^9 = O(0,2,0) = (0,0,2)$. Then, we can see that the $gcd(p) = gcd(p^9) = 2$.

**Definition 3.7.** $m_p = min\{k \,|\, p_1^k = 0\}$.

$m_p$ in the example given below definition 3.6 would be 6 since $p^6 = (0,2,6)$ and 6 is the smallest k such that $p_1 = 0$.

**Definition 3.8.** $M(p) = min\{k \,|\, p_1^k = p_2^k = 0\}$.

$M_p$ in the example given below definition 3.6 would be 9 because 9 is the smallest k such that both $p_1$ and $p_2$ are zero.

**Definition 3.9.** $h(p) = |p^{m(p)}|$

Note that the magnitude sign in this case refers to the notation used in defintion 3.5.

**Definition 3.10.** $f(x,y,z) = 1/2(x+y+z-2gcd(x,y,z)+h(x,y,z))$ where x,y,z are nonnegative integers.

Along with the previous definitions, a quick proof of the following theorem regarding periodic words is important to understand the proof of Fine and Wilf's theorem.

**Theorem 3.11.** *Given $p \in \mathbb{Z}$ with $p \geq 0$, a word $w$ with $|w| = n$ has period $p$ if and only if* $\mathrm{pref}_{n-p}(w) = \mathrm{suff}_{n-p}(w)$.

*Proof.* Let us prove the first part of the theorem, which is: if $\mathrm{pref}_{n-p} = \mathrm{suff}_{n-p}$, w has period p. Let $w = a_1 a_2 \ldots a_n$ having $a_i$ as a letter $\in$ A of alphabets over w. By the equality $\mathrm{pref}_{n-p} = \mathrm{suff}_{n-p}$, $a_1 = a_{n-(n-p)+1} = a_{p+1}$. Similarly $\forall i \,|\, 0 \leq i \leq n$, $a_i = a_{n-(n-p)+i}$, which is the definition of a word having period p. The second part of the theorem is if w has period p, $\mathrm{pref}_{n-p} = \mathrm{suff}_{n-p}$. Let $w = a_1 a_2 \ldots a_p a_{p+1} a_{p+2} \ldots a_{2p} \ldots a_{kp+1} \ldots a_{kp+j}$ where $k = \lfloor |w|/p \rfloor$ and $j = |w| \pmod p$ satisfying $j \leq p-1$. $\mathrm{pref}_{n-p}(w) = a_1 a_2 \ldots a_{k(p-1)+j}$. $\mathrm{suff}_{n-p}(w) = a_{p+1} a_{p+2} \ldots a_{kp+j} = a_1 a_2 \ldots a_{k(p-1)+j}$. Then, we can observe that $\mathrm{pref}_{n-p} = \mathrm{suff}_{n-p}$. ∎

**Lemma 3.12.** *Let $p, q \in \mathbb{Z}^+$ satisfying $p < q$. Let $w$ be a word such that $|w| = n$ and has two periods p and q. Then, the prefix and suffix of w of length n-p has period q-p.*

*Proof.* Let $w = a_1 a_2 \ldots a_n$, $w_k = \mathrm{pref}_{n-p}(w)$, and $(n-p) \geq (q-p)$. For the case $(n-p) < (q-p)$, lemma is true trivially. By theorem 3.11, if we show $\mathrm{pref}_{q-p}(w_k) = \mathrm{suff}_{q-p}(w_k)$, then the proof is complete. $\mathrm{pref}_{q-p}(w_k) = a_1 a_2 \ldots a_{q-p}$ is also a prefix of w, so $\mathrm{pref}_{q-p}(w_k) = \mathrm{suff}_{q-p}(w) = a_{n+p-q+1} \ldots a_n$. Because w has period of p, $\mathrm{suff}_{q-p}(w) = a_{n-q+1} \ldots a_{n-p}$. This shows that $\mathrm{pref}_{q-p}(w_k) = \mathrm{suff}_{q-p}(w_k)$ because suffix of w of length $\leq n-p$ is also suffix of $w_k$. By symmetry, $\mathrm{suff}_{n-p}(w)$ also has period q-p. ∎

**Lemma 3.13.** *Let $w$ be a word with $|w| = n$ having periods $p$, $q \in \mathbb{Z}^+$. Let $j = \mathrm{pref}_p(w)w$. $j$ has periods $p$ and $p$+$q$.*

*Proof.* By construction, j has period p. Let $p + q = p'$ and $|w| > p'$ because the other case is trivial. $\mathrm{suff}_{|j|-p'}(j) = \mathrm{suff}_{|w|-(p'-p)}(w) = \mathrm{pref}_{w-(p'-p)}(w) = \mathrm{pref}_{|j|-p'}(j)$ because w has period $p' - p = q$ and $|j| - p' = n - q < n$. By theorem 3.11, j has period $p'$. ∎

Now that we have covered important terms and lemmas for Fine and Wilf's theorem, we will prove the main theorem of this section now.

**Theorem 3.14.** *Let $w$ be a word with periods $p_1, p_2, p_3$ satisfying $p_1 \leq p_2 \leq p_3$. If $|w| \geq f(p_1, p_2, p_3)$, $w$ also has the period $gcd(p_1, p_2, p_3)$.*

*Proof.* Let $w = uv, |u| = p_1, |v| = |w| - p_1$. By lemma 3.12, v has periods $p_1, p_2 - p_1, p_3 - p_1$. $|v| = |w| - p_1 \geq f(p_1, p_2, p_3) - p_1 = 1/2(p_1 + p_2 - p_1 + p_3 - p_1 - 2gcd(p_1, p_2, p_3) + h(p_1, p_2, p_3) = f(p_1, p_2 - p_1, p_3 - p_1)$ because $gcd(p_1, p_2, p_3) = gcd(p_1, p_2 - p_1, p_3 - p_1)$ and same holds for $h(p_1, p_2, p_3)$ because of Euclidean algorithm. Applying this process recursively to v, we ultimately end with up the triplet $(0, 0, gcd(p_1, p_2, p_3))$. Note that once $p_1^k = 0$, we begin working with $p_2$ and $p_3$. Now applying lemma 3.13 in the reverse direction, we arrive at the conclusion of w with period $gcd(p_1, p_2, p_3)$. ∎

Note that when we plug in 0 for $p_1$ in the statement of theorem 3.14, we obtain $f(0, p_2, p_3) = 1/2(p_2 + p_3 - 2gcd(p_2, p_3) + p_2 + p_3) = p_2 + p_3 - gcd(p_2, p_3)$, and the necessary condition of classic Fine and Wilf's theorem is satisfied.

## 4. Sturmian Word

Let's remind our self the definition of a Sturmian Word, which is

**Definition 4.1.** A word $w$ is called a Sturmian word if $|w|$ is infinite, is composed of only two distinct alphabets, and contains $n + 1$ factors for each $n \geq 0$.

In the introduction section, we saw an example of a Sturmian Word, which was Fibonacci word, a word formed by repeatedly concatenating two previous terms of a sequence. In this section, we will look at seemingly different definitions that are equivalent to each other and prove this relationship. Note that the type of words we observe in this section are composed of only two alphabets, and we will note those two by using $\{0, 1\}$. Below is the list of all equivalent definitions of a Sturmian word:

(i) A word s is Sturmian

(ii) A word s is balanced and aperiodic

(iii) A word s is irrational mechanical

The proofs of these equivalent definitions will appear in the later part of this section as some basic definitions and examples are necessary to go over beforehand. The definition of complexity function allows us to formally define a Sturmian word, which is expressed through the equation below

$$P(x, n) = Card(F_n(x)) \tag{4.1}$$

For example, let word w be "nikash", $P(w, 3) = 4$. Using definition 4.2, let's redefine Sturmian word using the complexity function.

**Definition 4.2.** A word w is Sturmian if $|w|$ is infinite, is over 2 alphabets, and $P(w, n) = n + 1$.

Note that we can see that a Sturmian word must have a length greater than 1 because $P(w, 1) = 2$. We will begin the first proof of this section by proving how the Fibonacci word 0100101001001... is Sturmian. Before that, we will begin with a few definitions and examples to make more sense of the proof of Fibonacci number being Sturmian.

**Definition 4.3.** Let $\phi$ be defined as a morphism on the set $\{a, b\}$ such that $\phi(a) = ab$ and $\phi(b) = a$.

**Definition 4.4.** A right special factor of a word x is a word u such that u0 and u1 are factors of x.

Using the definition of right special factor, we can redefine the definition of Sturmian word as having exactly one special right factor of each length. This is in fact a theorem that is an if and only if condition. We will prove this theorem before diving into the proof of Fibonacci word being Sturmian as the proof of the theorem below will aid us in understanding the proof of fibonacci word satisfying Sturmian conditions.

**Theorem 4.5.** *The word u is Sturmian if and only if it has only exactly one right special factor of each length n > 0.*

*Proof.* Let's suppose the word u has period p for some p $\in \mathbb{Z} \,|\, p \geq 0$. Then, for a factor of u of length p has at most p factors since $a_i = a_{i+p}$, this contradicts how we defined u as a Sturmian word. Thus, we can see that u cannot be periodic. Because u is not periodic, if we randomly select a nonnegative integer k, let set $X = \{a_k a_{k+1} \dots a_{k+j-1}, a_{k+1} a_{k+2} \dots a_{k+j}, \dots a_{k+j-1} a_{k+j} \dots a_{k+2j-2}\}$ must be pairwise unique for some $j \in \mathbb{Z} \,|\, j > 0$. Because of the necessity in this property in u, if u is sturmian, there has the be one right special factor of each length j so that $|X| = j + 1$ to satisfy the Sturmian property of the word. If the word has exactly one right special factor of each length j, that means for all $j > 0$, the word is not periodic (because the relationship $a_i = a_{i+p}$ would not hold), and set X mentioned in the first part of the proof would have the carnality of j+1. ∎

This morphism can be used in defining the Fibonacci number. Because the number is formed by the operation $\phi^\omega(0)$ with 0 representing a and 1 representing 0 in definition 4.4. The notation $\phi^\omega(0)$ here refers to infinitely applying morphism $\phi$ on 0. To give an example of how this would work, $\phi(0) = 01$. Then, $\phi(01) = \phi(0)\phi(1) = 010$. $\phi(010) = \phi(0)\phi(1)\phi(0) = 01001$. This process would be applied endlessly to form the fibonacci word.

**Proposition 4.6.** *Let w represent the Fibonacci word defined as $\phi^\omega(0)$. Then, $P(w, n) = n + 1\, \forall n \in \mathbb{Z}, n \geq 0$.*

*Proof.* Note that the Fibanacci word w has the property $\phi(w) = w$ because of the way it is defined. Because of this, w can be described as product of words "01" and "0". So, "11" is not a factor of w. The word "000" is also not a factor of w because if "000" were a factor of w, "000" needs to be a factor of some prefix of a factor of w. And, the prefix of w would need to begin with "11" as $\phi(11) = 00$. Thus, 000 cannot be a factor of w. In this fashion, we will prove the fibonacci number is Sturmian using induction. Let $f_{-1} = 1, f_0 = 0$ for convenience. To prove the Fibonacci word is Sturmian, we need to prove it has exactly one right special factor of each length $n \geq 0$(refer to the proof of theorem 4.5). Let us first prove that for no word x, both 0x0 and 1x1 are factors of w. If x is a neutral element, then 11 cannot be a factor. If x is a single letter, then 111 cannot be a factor and 000 cannot be a factor, so there is no case in which both 0x0 and 1x1 are factors. Let's assume that both 0x0 and 1x1 are factors of w. In order for this to be true, the relation

$$x = 0y0 \tag{4.2}$$

needs to be satisfied for some word y because the word 1x1 needs to be a factor of w. By plugging in equation 4.2 into 0x0 and 1x1, we get that 00y00 and 10y01 are factors of w. We know that because $\phi(w) = w$, $\exists$ factor z of f such that $\phi(z) = 0y$. Specifically, $\phi(1z1) = 00y0$, $\phi(0z0) = 010y01$. Since $|z| \leq |\phi(z)| < |x|$, this is a contradiction because x should cover all words in the form of 0x0 and 1x1. Furthermore, we first show that w has at most one right special factor by arbitrarily assigning u and v as right special factors of same lengths. Let x be the longest common suffix of u and v. Then, four words 0x0, 0x1, 1x0, and 1x1 are factors of w, so this contradicts our proof that the factors of the form 0x0 and 1x1 cannot both exist. Let us now prove that f has at least one right special factor of each length. To prove this, let's use the relation

$$w_{n+2} = g_n \tilde{w}_n \tilde{w}_n t_n \tag{4.3}$$

for n $\geq$ 2 where $g_2 = \epsilon$. For $n \geq 3$,

$$g_n = w_{n-3} w_{n-4} \dots w_0 \tag{4.4}$$

$$t_n = \begin{cases} 01 & \text{if n is odd} \\ 10 & \text{if otherwise} \end{cases}$$

Because the first letter of $\tilde{f}_n$ is opposite of the first letter of $t_n$, we can observe that for any $n \geq 2$, $\tilde{f}_n$ is a right special factor. Since suffix of any right special factor is also a right special factor, we have proven that there is a right special factor of any length. ■

The proof of the equation 4.3 comes from induction that $\phi(\tilde{u})0 = 0\tilde{\phi(u)}$. Based on this relation, we can observe that $\phi(\tilde{f}_n t_n) = 0 f_{n+1} t_{n+1}$ and $\phi(g_n)0 = g_{n+1}$. Based on this, equation 4.3 is constructed. Now that we have developed more understanding for the first definition of a Sturmian word, which is a word having $n + 1$ factors of length n, we will cover the second equivalent definition, which states that a Sturmian word is aperiodic and balanced. Note that when we were trying to prove theorem 4.5, we already proved the fact that a Sturmian word is aperiodic. So, we only need to prove a Sturmian word is balanced.

**Definition 4.7.** Given a word x, the height of word, also denoted as $h(x)$, is the number of 1's in x

To give a quick example, if x is the word"1011", $h(x) = 3$. Arising from definition 4.7, we can also define another important term called balance.

**Definition 4.8.** Given two words x,y of the same length, their balance $\delta(x, y)$ is a number $\delta(x, y) = |h(x) - h(y)|$.

Thus, the balance $\delta(1011, 1000) = |3 - 1| = 2$. From the definition above, we can define a critical term when we describe a set balanced.

**Definition 4.9.** A set X is balanced if for x,y $\in$ X, with $|x| = |y|$, $\delta(x, y) \leq 1$.

**Definition 4.10.** An infinite or finite word is balanced if the set of all the factors of the word is balanced.

Based on the three previous definitions, we will prove the implications of a balanced word on the number of factors.

**Proposition 4.11.** *Let X be the set of all factors of a word. If X is balanced, for all* $n \geq 0$, $Card(X \cap A^n) \leq n + 1$.

**Proposition 4.12.** *Let X be a set of factors of a word. X is unbalanced if and only if there exists palindrome word w such that both 0w0 and 1w1 are in X.*

We will skip over the proofs of proposition 4.11 and proposition 4.12 so that we don't go too much into the details for the proof of (i) and (ii) in definition 4.1 being the equivalent definitions. But, the proofs require the use of contradictions just like the one displayed in the proof of proposition 4.6. So, the proofs of proposition 4.11 and 4.12 are left as exercises for readers. Using the fact that both proposition 4.11 and proposition 4.12 are true, we can arrive to the conclusion that the set of factors of w, the Fibonacci number, is balanced because the words of the form 0w0 and 1w1 do not exist as factors of w. Then, using proposition 4.11, there are factors of length n less than or equal to $n + 1$. Also, aperiodicity means there are at least $p+1$ factors of any factor of length p. Because we have the inequality in both direction for the number of factors of length n$(k \geq n + 1, k \leq n + 1)$ where k is the number of factors of length k, we have the relation

$$k = n + 1. \tag{4.5}$$

We can now formally prove the statement

**Theorem 4.13.** *If an infinite binary word w is aperiodic and balanced, then w is Sturmian.*

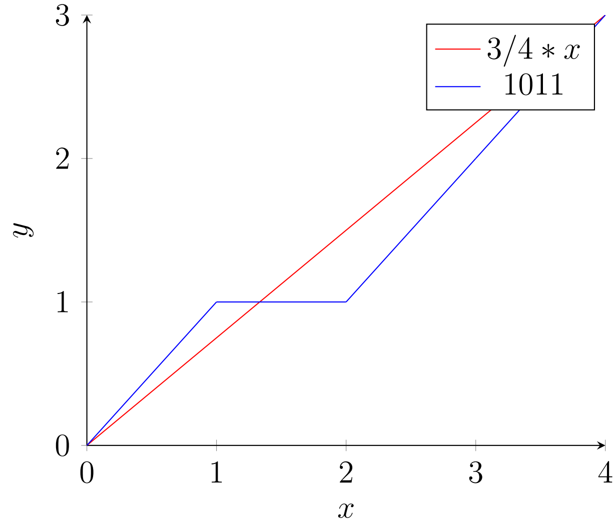
This theorem states that part (i) in definition 4.1 implies part (ii) in definition 4.2.

*Proof.* From proposition 4.11 and proposition 4.12, balanced word means $P(w, n) \leq n + 1$. Aperiodicity means $P(w, n) \geq n + 1$ for all n. Because of those two inequalities, w is Sturmian. ∎

The proof in the reverse direction uses the assumption that if w is Sturmian and unbalanced, w is eventually periodic. Now that we have proven the equivalnce of definition (i) and (ii) in definition 4.1, we will prove the equivalence of defintion (iii) with (i) and (ii). To do that, let's define a few important terms.

**Definition 4.14.** The slope of a nonempty word w is the number $\pi(x) = h(x)/|x|$.

Recall that $h(x)$ in the definition above refers to the height of the word defined in definition 4.7. To give an example, slope of the word 1011 would be 3/4 because $h(x) = 3$ and $|x| = 4$. The slope of a word can be visually represented in a graph.




We see that the blue line segments represent the piece-wise increase of the word "1011" where each interval of the x-axis is the single character of the word in successive order, and the y-axis is the number of accumulated 1's in the word. When the entire word "1011" is examined, we see that the blue line concurs with the red line, which is $y = 3/4 * x$ with the domain being $x : [0, 4]$. Thus, we can see that the slope of the red line, 3/4, is equal to the slope of the word "1011". We can also verify the relation

$$\pi(xy) = |x|\pi(x)/|xy| + |y|\pi(y)/|xy| \tag{4.6}$$

because the relation above equals $(h(x) + h(y))/|xy| = h(xy)/|xy| = \pi(xy)$. Based on the slope of the word, we will introduce the balance of the word in terms of slope by proving the following theorem

**Theorem 4.15.** *A factorial set of words X is balanced if and only if, for all x,y $\in$ X, x,y $\neq \epsilon$, $|\pi(x) - \pi(y)| < 1/|x| + 1/|y|$.*

Actually, Lothaire's book on the proof going in the direction from the inequality being satisfied to X is balanced is unclear to me, so we will only prove the proof going from X is balanced to satisfying the inequality.

*Proof.* Let us suppose X is balanced $|x| > |y|$. Let us express $x = zt$ with $|z| = |y|$. By induction on $|x| + |y|$, we can show that $|\pi(t) - \pi(y) < 1/|t| + 1/|y|$. This inequality is true because the strong induction on $|x| + |y|$ allows us to assume the inequality relation in the theorem statement to be true as $|t| + |y| < |x| + |y|$. Because the set X is a factorially balanced set, $|h(z) - h(y)| \leq 1$ and $|\pi(z) - \pi(y)| \leq 1/|y|$. By equation 4.6, $\pi(x) - \pi(y) =$

$\pi(zt)-\pi(y) = |z|/|x|*\pi(z)+|t|/|x|*\pi(t)-\pi(y) = |z|/|x|(\pi(z)-\pi(y))+|t|/|x|(\pi(t)-\pi(y)$. Thus, $|\pi(x)-\pi(y)| < 1/|x|+|t|/|x|(1/|y|+1/|t|) = 1/|x|+1/|y|$. For the case $|x| = |y|$, $\pi(x)-\pi(y) = (h(x)-h(y))/|x| < 2/|x|$ because $h(x)-h(y) < 1$ by the definition of X being a factorially balanced set. ■

The inequality relation at the very end of the first part of the proof in the case of $|x| > |y|$ is sufficed from the two previous inequality relations we formed from the strong induction and the definition of X being balanced. Note that some algebra is required to show the equality at the end, which we will show here. Just for clarity, we will restate the equality here

$$1/|x| + |t|/|x|(1/|y| + 1/|t|) = 1/|x| + 1/|y|. \tag{4.7}$$

$1/|x| + |t|/|x|(1/|y| + 1/|t|) = 1/|x| + (|x| - |y|)/|x|(1/|y| + 1/(|x| - |y|) = 1/|x| + (1 - |y|/|x|)(1/|y| + 1/(|x| - |y|) = 1/|x| + 1/|y| + 1/(|x| - |y|) - 1/|x| - |y|/(|x|(|x| - |y|)) = 1/|y| + 1/(|x| - |y|) - |y|/(|x|(|x| - |y|)) = 1/|y| + 1/|x|$. We will now introduce when the slope of a word is irrational.

**Proposition 4.16.** *Let $x$ be an infinite balanced word. The slope $\alpha$ of $x$ is a rational number if and only if $x$ is eventually periodic.*

*Proof.* Let $x = uy^\omega$. $\pi(u * y^n) = (h(u) + nh(y))/(|u| + n|y|)$. $\pi(x) = \lim_{n\to\infty} \pi(uy^n) = h(y)/|y| = \pi(y)$. Thus, if x is eventually periodic, the slope $\alpha$ of x is a rational number. ■

Before proving the proposition in the other direction, let us introduce two inequalities that are satisfied within the values $\alpha$, the slope of a word, u, a factor of a word.

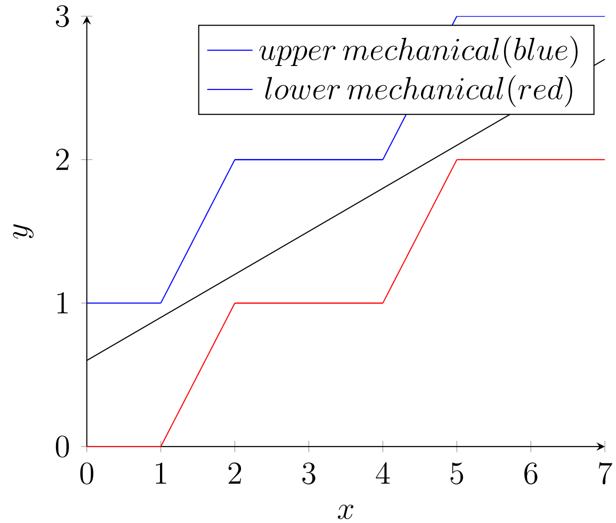
$$\alpha * |u| - 1 < h(u) \le \alpha * |u| + 1 \tag{4.8}$$

$$\alpha * |u| - 1 \le h(u) < \alpha * |u| + 1 \tag{4.9}$$

However, the inequalities 4.8 and 4.9 are strict if $\alpha$ is irrational. Let us continue with the other direction of proof of proposition 4.16.

*Proof.* Let us assume (4.8) holds as the case in which (4.9) makes the same argument due to symmetry. The slope $\alpha$ of x is a rational number such that $\alpha = p/q$ where $(p, q) = 1$. Thus, any factor of length q has the height of p or p+1. There is a finite number of factors of length q with the height of p or p+1 because otherwise there is a factor w = uzv of x such that $|u| = |v| = q$ and $h(u) = h(v) = p + 1$. $h(uzv) = 2 + 2q + h(z) \le 1 + \alpha * q + \alpha * |z| + \alpha * q = 1 + 2p + \alpha|z|$. Thus, $h(z) \le \alpha * |u| - 1$, which contradicts (4.8). From this, we can observe that x can be factored in the form $x = ty$ such that every factor of length q of y has the same height. We can now consider a factor in of x in the form azb with the length of q+1 and a,b being letters such that $h(az) = h(zb)$, which means a = b. This shows that y is periodic with the period p. Thus, x is eventually periodic. ■

Now that we have gone over the meaning of rational slope in the word, we will explore the definitions of mechanical, which is the second part of the definition (iii) of definition 4.1. Given two real numbers $\alpha, \rho$ with $0 \le \alpha, \rho \le 1$ with $s_{\alpha,\rho} : \mathbb{N} \longrightarrow A$ $s'_{\alpha,\rho} : \mathbb{N} \longrightarrow A$ $s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + p \rfloor$ $s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + p \rceil$ for $n \ge 0$. We can observe that both $s_{\alpha,\rho}$ and $s'_{\alpha,\rho} \in \{0, 1\}$. Here, $s'_{\alpha,\rho}$ is called the upper mechanical word, and $s_{\alpha,\rho}$ is called the lower mechanical word. Below is an example of how these words would be illustrated on a graph.




The integer coordinates below the black line in the figure can be represented with the expression $(n, \lfloor \alpha * n + \rho \rfloor)$, which are the integral coordinates in the red line segments. Similarly, the integer coordinates above the black line are represented by the expression $(n, \lceil \alpha * n + \rho \rceil)$. Now, we will introduce an important term in the slope of the word.

**Definition 4.17.** A mechanical word is irrational or rational according to the rational or irrational state of its slope.

Let us now prove the three definitions listed at the beginning of this section for Sturmian word are equivalent. Here are the restatements of equivalent definitions:

**Definition 4.18.** (1)A word s is Sturmian

**Definition 4.19.** (2)A word s is balanced and aperiodic

**Definition 4.20.** (3)A word s is irrational mechanical

Note that we have proven that definition 4.18 is equivalent to definition 4.19. So, we would need to prove definition 4.19 is equivalent to definition 4.20, and we will be done with proving all three definitions are equivalent. Before we prove the equivalence between definition 4.19 and definition 4.20, we will prove two lemmas. In proving those two lemmas, we will make use of the equation below.

$$x' - x - 1 < \lfloor x' \rfloor - \lfloor x \rfloor < x' - x + 1 \tag{4.10}$$

**Lemma 4.21.** *Let $s$ be a mechanical word with slope $\alpha$. Then $s$ is balanced of slope $\alpha$. If $\alpha$ is rational, $s$ is periodic. If $\alpha$ is irrational, $s$ is aperiodic.*

*Proof.* Let $s = s_{\alpha,\rho}$be a lower mechanical word. Symmetry allows the same proof for when s is an upper mechanical word. $h(u) = \lfloor \alpha*(n+\rho)+\rho \rfloor - \lfloor \alpha*n+\rho$. So, $\alpha*|u|-1 < h(u) < \alpha*|u|+1$. Which means $\lfloor \alpha * |u| \rfloor \leq h(u) \leq \lfloor \alpha * |u| \rfloor + 1$. Since $h(u)$ can only take on two values, and by inequality established previously, $\pi(u) - \alpha < +1/|u|$. We can now show that as the slope of u approaches $\alpha$, $|u|$ approaches infinity. This proves s is balanced. By the proof of proposition 4.16, if $\alpha$ is irrational, s is aperiodic. If $\alpha$ is rational, s is periodic by the same proposition. ∎

Let us move on to the proof of second lemma, which will prove that all three definitions of Sturmian word are equivalent.

**Lemma 4.22.** *Let $s$ be a balanced infinite word. If $s$ is aperiodic, then $s$ is irrational mechanical. If $s$ is purely periodic, then $s$ is rational mechanical.*

The author will leave the proof of this lemma as an exercise. But, the proof requires the use of proofs of other propositions, theorems, and lemmas in this section of paper. Thus, we have established the equivalence relationships between the three definitions of Sturmian established in definition 4.1. I hope you enjoyed learning about properties of Sturmian words!

## 5. Plactic Monoid

In this section, I will introduce Young Tableaux and Schensted's algorithm to present the ideas of plactic monoid eventually. The idea behind plactic monoid is that the tableau algorithm of finding the longest nondecreasing subword of a given word leads to plactic monoid. This observation was made by D. Knuth, while the algorithms were invented from A. Tableau and Schensted. I hope you enjoy learning about one of the major algorithm(RSK) in combinatorics and how that leads to plactic monoid!

As a convention, we will use A as a set of totally ordered alphabet of n letters in the order $a_1 < a_2 \ldots < a_n$. We will denote $A = \{1, 2, \ldots, n\}$. We will first introduce an important combinatorial structure called Young tableau, something that comes from group theory. Let us define a few important terms to learn what a tableau is.

**Definition 5.1.** A nondecreasing word v $\in A^*$ is called a row.

For example, the factor "abcd" of "abcdcb" would be a row.

**Definition 5.2.** Let u and v be two rows such that $u = x_1 x_2 \ldots x_r$ and $v = y_1 y_2 \ldots y_s$ with $x_i, y_i \in A$. We claim u dominates v if $r \leq s$ and for all $i = 1, 2, 3, \ldots r$, $x_i > y_i$.

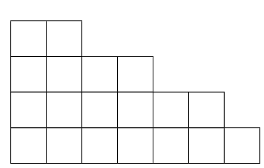
A tableau is a planar object that represents word w as the left justified superposition of its rows. So, we can represent t=68 4556 23357 1112444 with $A = \{1 < 2 < \ldots\}$,

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 8 | | | | | |
| 4 | 5 | 5 | 6 | | | |
| 2 | 2 | 3 | 3 | 5 | 7 | |
| 1 | 1 | 1 | 2 | 4 | 4 | 4 |

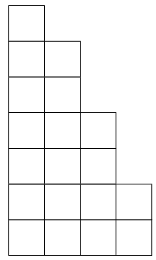
Here, we can notice that each column of the young diagram above is a strictly decreasing sequence of numbers.

**Definition 5.3.** A strictly decreasing word is called a column.

Also, reading from the bottom row to the top row, the lengths of the rows form a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \ldots \geq \lambda_k)$, and this sequence is called the shape of t. We call this sequence a partition of the integer $|\lambda|$. The example $\lambda = (7, 6, 4, 2)$. The graphical interpretation of this partition is called the Ferrers diagram. We can represent the Ferrers diagram of (7,6,4,2) as



The conjugate partition $\lambda'$ of $\lambda$ is obtained by reading the heights of the columns from left to right. The conjugate partition of (7,6,4,2) is (4,4,3,3,2,2,1), which is represented by



Schensted's algorithm associates a tableau $t = P(w)$ with a word w $\in A^*$. Given a letter x and row v, one can insert x into v by applying $P(vx) = vx$ if vx is a row, and $P(vx) = y_i v'$ otherwise. $y_i$ is the leftmost letter in v that is greater than x, and v' is obtained from replacing the letter $y_i$ with v. The insertion of letter x into a tableau $t = v_1 v_2 \dots v_k$ involves inserting x into the bottom row $v_k$ and if $v_k x$ is not a row, then $y_i v_k'$ is formed, then $y_i$ is inserted into row $v_{k-1}$ and so on until the algorithm reaches the first row or all the letters are placed in the right rows. More formally, this process can be written recursively as

$$P(tx) = \begin{cases} tx & \text{if } v_k x \text{ is a row} \\ P(v_1 \dots v_{k-1} y) v_k' & \text{if } P(v_k x) = y v_k' \end{cases}$$

Schensted's algorithm can be shown through an example below where we construct a tableau for the word "132541" going from top to bottom and left to right.

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|cc}\cline{1-1} 3 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline\end{array}$$

$$\begin{array}{|c|c|c}\cline{1-2} 3 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \quad \begin{array}{|c|cc}\cline{1-1} 3 & & \\ \cline{1-2} 2 & \multicolumn{1}{c|}{5} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

Below, we have an interesting theorem regarding a tableau of a word, which states

**Theorem 5.4.** *The maximal length of a nondecreasing subword of word w is equal to* $|v_k|$ *or the length of the last row of the tableau.*

Similarly, the maximal length of the decreasing sequence is equal to the height of the first column of a tableau diagram. As an example, the longest nondecreasing sequence of "132541" has the length 3 in which one of them is "125" as the length of the bottom row of its tableau is also 3. Before we get into learning what plactic monoid is, let's explore the equivalence relation in Schensted tableau.

**Definition 5.5.** $l_k(w)$ is the maximum of the sum of the lengths of k disjoint nondecreasing subwords of w.

**Definition 5.6.** $l_k(w)'$ is the maximum of the sum of the lengths of k decreasing subwords of w.

Let's remind our self that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the shape of P(w) and $\lambda' = (\lambda'_1, \ldots, \lambda'_s$ is the conjugate partition.

**Theorem 5.7.** *For $k = 1, \ldots, r$, $\lambda_k = l_k(w) - l_{k-1}(w)$ and for $k = 1, 2, \ldots, s$, $\lambda'_k = l_k(w)' - l_{k-1}(w)$.*

To understand the proof of theorem 5.7., we need to understand the equivalence relation in Schensted tableau. Let's define the equivalence relation below.

**Definition 5.8.** $u \sim v \longleftrightarrow P(u) = P(v)$.

We can observe that if a word has length $\leq 2$, then $u \sim v$ implies $u = v$. The first major case appears when the length of the word is at 3. Examine the four equivalence relationships below. $P(xyz) = P(zxy) = \begin{array}{|c|c|}\cline{1-1} z & \multicolumn{1}{c}{} \\ \hline x & y \\ \hline\end{array}$ $P(yzx) = P(yxz) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & z \\ \hline\end{array}$ $P(xyx) = P(yxx) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & x \\ \hline\end{array}$ $P(yxy) = P(yyx) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & y \\ \hline\end{array}$ Let us now explore the definition of plactic monoid, which is the main purpose of this section of the paper.

**Definition 5.9.** The plactic monoid on the alphabet A is the quotient $Pl(A) = A^*/ \equiv$, where $\equiv$ is the congruence generated by the Knuth relations $xzy \equiv zxy (x \leq y \leq z)$, $yxz \equiv yzx (x < y \leq z)$

In fact, plactic monoid represents an equivalence relationship between two different words in lexicographic order and expresses them as being equivalent based on the equivalence of Schensted tableau of the word. From here, we can go on to prove Green's theorem, which states the interpretations of the lengths of rows and columns of the young diagram of a word, which leads us to prove theorem 5.4. because this theorem is merely a sub case of Green's plactic invariants. This section was meant to be more introductory relative to other two main sections of this paper which were on Fine and Wilf's theorem and Sturmian word. So, this will be the end of this section. If you are interested in learning more about Knuth equivalences and other more interesting theorems related to plactic monoid, feel free to explore Lothaire's book on algebraic combinatorics, which is listed as the first item in references. I hope you enjoyed reading my paper! I will definitely produce more expository papers on other mathematical concepts and hopefully make my own research papers in the future. Thanks again!

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