Extending the Lebesgue Measure

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Basic Notions (1/2)

Definition (Algebra)

A collection $\mathcal{A}$ of subsets of a set $\mathcal{X}^a$ is an algebra provided that (1) $\emptyset \in \mathcal{A}$, (2) if $A \in \mathcal{A}$ then its complement is in $\mathcal{A}$, and (3) a finite union of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

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A finitely additive measure is a function, μ from an algebra to the set, $[0, \infty]$, satisfying: (1) $\mu(\emptyset) = 0$ and (2) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint A, B in the algebra.

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Definition (σ algebra)

An algebra is said to be a σ algebra if a countable union of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

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A finitely additive measure, μ, on a σ algebra is said to be countably additive if it satisfies $\mu(\bigcup E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for all $\mathbb{N}$ indexed sequences E_n, where the E_n belong to the σ algebra and are pairwise disjoint.

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Definition (Extension of measure)

An algebra $\mathcal{A}'$ with measure μ' is said to be an extension of the algebra $\mathcal{A}$ with measure μ, if $\mathcal{A} \subseteq \mathcal{A}'$ and the function μ' extends μ. An extension is said to be proper if $\mathcal{A}' \neq \mathcal{A}$.

The Problem of Measure

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For a subset of $\mathbb{R}^d$ what do we mean by its d-dimensional volume?

This question and its deep connection with the theory of the integral has been examined by several influential mathematicians of the 19th century including Augustin Cauchy, Lejeune Dirichlet, Bernhard Riemann, Camille Jordan, Emile Borel, Henri Lebesgue, and Giuseppe Vitali.

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- $m([0, 1]^d) = 1$.
- If E_n is a countable collection of pairwise disjoint sets in $\mathbb{R}^d$, then $m(\bigcup E_n) = \sum_{n=1}^{\infty} m(E_n)$. (Countable Additivity)
- If τ is an isometry[a] from $\mathbb{R}^d \to \mathbb{R}^d$ and E is a subset of $\mathbb{R}^d$ then $\tau(E)$ is measurable and $m(E) = m(\tau(E))$. (Invariance under isometries)

[a] An isometry is a distance preserving transformation

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Proof.

Assume for the sake of contradiction that such a measure, m existed. Partition $\mathbb{R}^d$ into equivalence classes using the relation: $x \equiv y \iff x - y \in \mathbb{Q}^d$. Using the Axiom of Choice, we can construct a set V containing exactly one element of each equivalence class's intersection with $[0, 1]^d$. Now pick an enumeration q_k of $([-1, 1] \cap \mathbb{Q})^d$, and then define $V_k := V + q_k$. One can easily verify that the sets V_k are pairwise disjoint and that $([0, 1] \cap \mathbb{Q})^d \subseteq \bigcup_k V_k \subseteq ([-1, 2] \cap \mathbb{Q})^d$, the latter implies that $1 \leq \sum_{k=0}^{\infty} m(V_k) \leq 3^d$, by translation invariance we have that for all k, $m(V_0) = m(V_k)$, therefore $1 \leq \sum_{k=0}^{\infty} m(V_0) \leq 3^d$, a contradiction. □

Not All Hope Lost

Even though one can not have such a measure on all subsets of $\mathbb{R}^d$, one can still hope to have such a measure on a "large" σ algebra $\subseteq \mathcal{P}(\mathbb{R}^d)$ that suffices for all practical purposes, and this is exactly what the Lebesgue measure is, on the σ algebra of Lebesgue measurable sets, $\mathcal{L}(\mathbb{R}^d)$. Therefore we have a notion of length, area, volume, and hyper-volume for all reasonable[1] sets.

[1] Informally speaking, any set that can be concretely defined is Lebesgue measurable

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Is there an extension of Lebesgue measure to a larger sigma algebra satisfying the dream properties? Is there a maximal such extension?

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Even though for all practical purposes the measure problem is solved, it is worth asking the following questions:

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The answer to both questions is **NO**, as we shall see in the next slide.

Are there countably additive, isometry invariant extensions of Lebesgue measure?(2/2)

That there is a countably additive, isometry invariant extension of Lebesgue measure was first proven by Edward Szpilrajn.

That there is no maximal countably additive isometry invariant measure was first proven by Harazivili in the one dimensional case. It was generalized to d-dimensions by Ciesielski and Pelc. The idea of the proof is to construct a special family of subsets in $\mathbb{R}^d$ $\{N_j, j = 0, 1, 2, 3, \cdots\}$, with $\mathbb{R}^d = \bigcup\{N_j, j = 0, 1, 2, 3, \cdots\}$, satisfying certain special properties, that make the existence of a maximal countably additive isometry invariant measure impossible.

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Therefore we now ask the following question: Is there a translation invariant, *finitely additive* extension of Lebesgue measure to all subsets of $\mathbb{R}^d$. Suprisingly the answer depends on the value of d as we shall see on the next slide.

Banach Tarski and the lack of thereof

Theorem (Banach Tarski)

For $d \geq 3$, it is possible to decompose the d-dimensional unit sphere into finitely many pieces and rearrange these pieces to form **two** *d-dimensional unit spheres.*

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Surprisingly in dimensions 1 and 2 it is possible to have a finitely additive, isometry invariant measure, measuring all subsets of $\mathbb{R}^d$ that extends the Lebesgue measure. Banach proved the $\mathbb{R}$ case using Hahn-Banach and extended it to the $\mathbb{R}^2$ case using an averaging trick. So there is no analogue of Banach Tarski for the line or the plane.

Is there something that makes Lebesgue the richest?

It is not unwise to ask if there is a property that makes Lebesgue measure the richest countably additive measure, the answer is **YES**, it is called regularity, indeed we have the following the theorem:

Theorem (Lebesgue is the maximal regular measure)

The Lebesgue measure is the maximal measure satisfying the dream properties and that for every $\epsilon > 0$ and measurable set, M there is an open set $\mathcal{O}$ such that the Lebesgue measure of $\mathcal{O} - M$ is less than ϵ

Thank You! Questions?