THE BANACH-TARSKI PARADOX

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Abstract. We present an exposition of the Banach-Tarski paradox and several related results. All of of the material presented here is classical. Our aim is to distill the key results and to provide short, clear proofs in an accessible manner.

Introduction

The Banach-Tarski paradox is a striking result in set-theoretic geometry. It is a result that strongly violates our physical intuition stating that (using the Axiom of Choice[1]) it is possible to cut up a ball into finitely many disjoint pieces and rearrange these pieces to form *two* balls. A version of the paradox first appeared in Hausdorff's book [Hau14]; although, the theorem first gained recognition following Banach and Tarski's paper [BT24]. Banach and Tarski proved a more general version of the paradox and laid the foundation for the more general notion of *paradoxicality*. Our aim is to not only provide an exposition of the Banach-Tarski paradox but also of the several extremely interesting results surrounding it. We shall divide this paper into three sections. In the first, we shall discuss the basics of the notions of equidecomposibility and paradoxicality. In the second, we shall prove the Banach-Tarski paradox in all dimensions greater than or equal to 3. Finally, we shall prove the non-existence of the paradox for the line and the plane and discuss the general notion of amenable groups.

1. Equidecomposibility and Paradoxicality

Recall that a group, $\mathcal{G}$ with product $*$, is said to act[2] on a set X, if for every $g \in \mathcal{G}$, we have a corresponding bijection from X to X, which we shall also denote by g, such that the bijection corresponding to the identity element in $\mathcal{G}$ is the identity bijection, and that for any $g, h \in \mathcal{G}$ and $x \in X$ we have $g(h(x)) = (g * h)(x)$. We shall require group actions in the definition of both equidecomposibility and paradoxicality.
We shall start with the definition of equidecomposibility, which codifies the idea of breaking and reassembling a set.

Definition 1.1 ($\mathcal{G}$-equidecomposibility)**.** Let $\mathcal{G}$ be a group acting on a set X, let A and B be subsets of X, then we say that A and B are $\mathcal{G}$-equidecomposable, which we shall denote as $A \equiv B$, if both A and B can be written as the disjoint union of finitely many subsets, $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, such that there exist $g_1, \ldots, g_n \in \mathcal{G}$ such that $g(A_i) = B_i$ for

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[1]The Axiom of Choice plays an important role in the proof of the Banach-Tarski paradox. Because every set can be measurable in Euclidean space without a large amount of choice(see [Sol70]), the Banach-Tarski paradox is independent of ZF. However, the paradox is strictly weaker than the full Axiom of Choice. Interestingly, it can be proven that the Banach-Tarski paradox holds for the rational ball even without choice.

[2]All our actions shall be taken to be faithful.

all i. If we have that B is $\mathcal{G}$-equidecomposable with a subset, C, of A then we shall write $A \succcurlyeq B$.
Note. Strictly speaking we should write $A \equiv_{\mathcal{G}} B$ and $A \succcurlyeq_{\mathcal{G}} B$; however, we shall slightly abuse notation as $\mathcal{G}$ should be clear from context.

It is easy to verify that $\mathcal{G}$-equidecomposibility is an equivalence relation.

Remark. The reader may notice a striking similarity between the notion of equidecomposibility and that of the so called notion of "congruence by dissection" for polygons in $\mathbb{R}^2$. The only seeming difference is that in congruence by dissection for polygons we don't care about the boundary segments; however, it turns out that for polygons in $\mathbb{R}^2$ these notions are equivalent. Indeed, two polygons in $\mathbb{R}^2$ are congruent by dissection if and only if they are equidecomposable. The forward direction can be shown by "absorbing" the troublesome boundary segment by an absorbtion argument(we will soon see similar arguments), the reverse direction is trivial.

For a fixed group $\mathcal{G}$ acting on the set X, one has the following important theorem:

Theorem 1.2 (Banach-Schröder-Bernstein)**.** *For all subsets A, B of X if $A \succcurlyeq B$ and $B \succcurlyeq A$ then $A \equiv B$.*

Proof. Recall the Banach mapping theorem[3], which states that for any two functions $f : A \to B$ and $h : B \to A$ there exist disjoint subsets A_1, A_2 of A and disjoint subsets B_1, B_2 of B such that $A_1 \cup A_2 = A, B_1 \cup B_2 = B, f(A_1) = B_1$ and $h(B_2) = A_2$. Now since A is $\mathcal{G}$-equidecomposable with a subset, C, of B and B is $\mathcal{G}$-equidecomposable with a subset, D, of A, one can easily see the existence of an injection $f : A \to B$ such that any subset of A is equidecomposable with its f-image and the existence of an injection $h : B \to A$ such that any subset of B is equidecomposable with its h-image, applying the Banach mapping theorem on the functions f, h one sees that A is $\mathcal{G}$-equidecomposable with B. ∎

We now turn to paradoxicality. The notion of paradoxicality codifies the intuitive idea of duplicating a set using certain actions on the set as is seen by the following definition and proposition:

Definition 1.3 ($\mathcal{G}$-paradoxicality)**.** Let $\mathcal{G}$ be a group acting on a set X, let E be a subset of X, then E is said to be $\mathcal{G}$-paradoxical if there exist disjoint subsets A, B of E such that $A \equiv B \equiv E$.

Proposition 1.4. *Let, $\mathcal{G}$ be a group acting on a set X, If $E \subseteq X$ is $\mathcal{G}$-paradoxical then there exist disjoint subsets, C, D of E such that $C \cup D = E$ and $C \equiv D \equiv E$.*

Proof. Since E is $\mathcal{G}$-paradoxical, there exist disjoint subsets A, B of E such that $A \equiv B \equiv E$. Now, let $C = A$ and $D = E \setminus A$ then by a simple application Banach-Shröder-Bernstein theorem we have that $A \equiv E \equiv E \setminus A$. ∎

Definition 1.5 (Paradoxical Group)**.** A group $\mathcal{G}$ is said to be paradoxical if it is $\mathcal{G}$ -paradoxical, where it acts on itself by left-multiplication.

Our next theorem allows us to "transfer" paradoxicality from a group to a set on which the group acts. It requires the Axiom of Choice to prove. This idea is central to the proofs of many paradoxical decompositions, including that of Banach-Tarski.

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[3]The Banach mapping theorem is a standard consequence of Knaster-Tarski's fixed point lemma.

Theorem 1.6. *If a group $\mathcal{G}$ is paradoxical and acts on a set X with no non-trivial fixed points(only the identity has fixed points) then X is $\mathcal{G}$-paradoxical.*

Proof. Using the Axiom of Choice one can construct a subset, I, of X, that consists of exactly one element from each $\mathcal{G}$-orbit. Given a subset S of $\mathcal{G}$, we shall let S^* be the set: $\{x \in X | x = s(i)$ for some $s \in S$ and $i \in I\}$, since X contains no non-trivial fixed points if A and B are disjoint subsets of $\mathcal{G}$, then A^*, B^* will be disjoint subsets of X. Now, since $\mathcal{G}$ is paradoxical there exist disjoint subsets A, B of $\mathcal{G}$ such that $A \equiv \mathcal{G} \equiv B$, we shall "push" this paradoxicality to X by considering A^*, B^*. Indeed, since we know that $A \equiv \mathcal{G}$ we can decompose A into disjoint sets, $A_1, \ldots, A_n$ such that there exist $g_1, \ldots, g_n \in \mathcal{G}$ with $\mathcal{G} = \bigsqcup_{i=1}^n g_i(A_i)$, now consider the corresponding disjoint decomposition of $A^* : A_1^*, ..., A_n^*$, since $\mathcal{G} = \bigsqcup_{i=1}^n g_i(A_i)$, we have that $X = \bigsqcup_{i=1}^n g_i(A_i^*)$, and therefore $A^* \equiv X$, we can do the same thing with B^* to get that $A^* \equiv X \equiv B^*$, therefore X is $\mathcal{G}$-paradoxical. ∎

Corollary 1.7. *If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $\mathcal{H}$ is paradoxical then $\mathcal{G}$ is paradoxical.*

Proof. The group $\mathcal{H}$ acts on the group $\mathcal{G}$ with no non-trivial fixed points therefore by 1.6, $\mathcal{G}$ is $\mathcal{H}$-paradoxical which means that it is also $\mathcal{G}$-paradoxical. ∎

One also has the following converse to 1.6:

Theorem 1.8. *If a group $\mathcal{G}$ acts on a non-empty set X that is $\mathcal{G}$-paradoxical then $\mathcal{G}$ is paradoxical.*

Proof. We will show that for every $x \in X$ the $\mathcal{G}$-orbit of x, $\mathcal{G}_x$, is $\mathcal{G}$-paradoxical, the non-emptyness of X would then imply that $\mathcal{G}$ is paradoxical. Fix $x \in X$, since X is $\mathcal{G}$-paradoxical, there exist pairwise disjoint subsets of X, $A_1, \ldots, A_n, B_1, \ldots, B_m$ and elements of $\mathcal{G}$, $g_1, \ldots, g_n, h_1, \ldots, h_m$, such that $X = \sqcup_{i=1}^n g_i A_i$ and $X = \sqcup_{i=1}^m h_i B_i$. For each A_i, let $A_i^* = A_i \cap \mathcal{G}_x$, given a $rx \in \mathcal{G}_x$, since $\mathcal{G}$ acts on X, there exists a y in some A_i such that $g_i y = rx$, solving for y, we get $y = g^{-1} rx$, therefore $y \in A_i^*$, and hence $\sqcup_{i=1}^n g_i A_i^* = \mathcal{G}_x$, similarly we have $\sqcup_{i=1}^n h_i B_i^* = \mathcal{G}_x$, and therefore $\mathcal{G}_x$ is $\mathcal{G}$-paradoxical. ∎

2. The Banach-Tarski Paradox

We shall begin this section with the notion of a free group on two generators, we shall use the free group's paradoxical nature to prove the Banach-Tarski paradox.

Definition 2.1 (Free group on two generators)**.** The free group on two generators is the group consisting of all reduced words in the language[4], $\mathscr{L} = \{a, a^{-1}, b, b^{-1}\}$. We shall denote the free group on two generators by F_2.

Proposition 2.2 (The free group on two generators is paradoxical)**.**

Proof. Let $S(x)$ denote the set of reduced words starting with x. Then we have that:

$$\begin{aligned} F_2 &= \{e\} \sqcup S(a) \sqcup S\left(a^{-1}\right) \sqcup S(b) \sqcup S\left(b^{-1}\right) \\ &= S(a) \sqcup aS\left(a^{-1}\right) \\ &= S(b) \sqcup bS\left(b^{-1}\right) \end{aligned}$$

Letting $A = S(a) \sqcup S\left(a^{-1}\right)$ and $B = S(b) \sqcup S\left(b^{-1}\right)$, we have that A and B are disjoint and that $A \equiv F_2 \equiv B$. Therefore F_2 is paradoxical. ∎

[4]A word is said to be reduced if it is in its most simplified form, so $a^{-1}ab$ is not reduced but b is.

Theorem 2.3 (SO(3) contains a copy of the free group on two generators)**.**

Proof. Let A denote the matrix representing a rotation by the angle $\arccos(1/3)$ about the x-axis, and let B denote the matrix representing a rotation by the angle $\arccos(1/3)$ about the z-axis. Then we have that:

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix}$$

$$B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

It can be shown by a tedious but not too difficult argument that no non-trivial reduced combination of the matrices A, A^{-1}, B, B^{-1} is equal to identity matrix, thereby showing that the matrices generate an isomorphic copy of the free group on two generators. ■

Remark. There is not much special about the particular example we gave, any sufficiently generic pair of rotations in SO(3) will generate a free group, see [Eps71].

Theorem 2.4 (Hausdorff's Paradox)**.** *There exists a countable subset, D, of the unit sphere $\mathbb{S}^2$, such that $\mathbb{S}^2 \setminus D$ is* SO(3)*-paradoxical.*

Proof. We know that there exists an isomorphic copy of F_2 which is a subgroup of SO(3), we might as well abuse notation slightly and denote this copy by F_2 as well. Consider the set of all points of $\mathbb{S}^2$ that remain fixed after an application of some non-trivial $f \in F_2$, denote this set by D, since F_2 is countable and every non-trivial rotation has exactly two fixed-points, D is also countable. It is not difficult to see that F_2 acts on the set $\mathbb{S}^2 \setminus D$, further it acts with no non-trivial fixed points, therefore since F_2 is paradoxical, by 1.6 we have that $\mathbb{S}^2 \setminus D$ is F_2-paradoxical which means that it is also SO(3)-paradoxical. ■

Our next theorem shows how we can absorb the countable set of points D, to show that the entire sphere $\mathbb{S}^2$ is SO(3)-paradoxical.

Theorem 2.5. *The set $\mathbb{S}^2 \setminus D$ is* SO(3) *equidecomposable with $\mathbb{S}^2$, therefore $\mathbb{S}^2$ is* SO(3)*-paradoxical.*

Proof. Since D is countable we can pick a line, ℓ, that goes through the origin and does not pass any of the points in D, another countability argument shows that there exists a rotation, ρ, about ℓ such that the sets $D, \rho(D), \rho^2(D), \ldots$ are pairwise disjoint. Let $\mathbf{D} = \bigcup_{i=0}^{\infty} \rho^i(D)$, then we have that $\rho(\mathbf{D}) = \mathbf{D} \setminus D$, therefore, since $\mathbb{S}^2 = \mathbb{S}^2 \setminus \mathbf{D} \sqcup \mathbf{D}$ and $\mathbb{S}^2 \setminus D = \mathbb{S}^2 \setminus \mathbf{D} \sqcup \rho(\mathbf{D})$, we have that $\mathbb{S}^2 \setminus D$ is SO(3) equidecomposable with $\mathbb{S}^2$. ■

We are now ready to prove the Banach-Tarski paradox.

Theorem 2.6 (Banach-Tarski)**.** *The unit ball, B^3, is* SO(3)*-paradoxical*

Proof. It suffices to prove that $B^3 \setminus \{\mathbf{0}\}$ is paradoxical, for then using the same method as in the previous theorem we can absorb the point $\mathbf{0}$. For each subset $S \subseteq \mathbb{S}^2$ consider the set, $S^* = \{cx | x \in \mathbb{S}^2, c \in (0,1]\}$, for disjoint subsets A, B of $\mathbb{S}^2$ we have that A^*, B^* are disjoint and it is not difficult to see that if A, B form a paradoxical decomposition of $\mathbb{S}^2$ then A^*, B^* will form a paradoxical decomposition of $B^3 \setminus \{0\}$. ∎

Using the paradox in 3 dimensions, one can obtain the paradox in all dimension greater than or equal to 3. As is shown by the following corollary:

Corollary 2.7. *The ball B^n is* $\mathrm{SO}(n)$*-paradoxical for all $n \geq 3$*

Proof. We shall prove this by induction, suppose for $n \geq 3$, the ball B^n is $\mathrm{SO}(n)$-paradoxical, now slice the set, $B^{n+1} \setminus \{\mathbf{0}\}$, into disjoint n-dimensional balls, it is not difficult to extend the paradoxicality of the n-dimensional balls to the set $B^{n+1} \setminus \{\mathbf{0}\}$, therefore $B^{n+1} \setminus \{\mathbf{0}\}$ is $\mathrm{SO}(n+1)$-paradoxical, which implies, as before, that B^{n+1} is $\mathrm{SO}(n+1)$-paradoxical. ∎

Using the Banach-Shröder-Bernstein theorem and 2.7 one has the following stronger form of the Banach-Tarski paradox:

Theorem 2.8. *If A and B are any two bounded subsets of $\mathbb{R}^n (n \geq 3)$, each having nonempty interior, then A and B are equidecomposable with respect to the isometry group, $\mathbb{E}^n$.*

Proof. We shall prove $A \succcurlyeq B$, for an exact imitation will get us $B \succcurlyeq A$. Pick balls A^* and B^* so that $A^* \subseteq A$ and $B \subseteq B^*$, now for some large enough number, k, B^* can be covered by k (non-disjoint) copies of A^*, if S is the union of k disjoint copies of A^* then by repeatedly applying Banach-Tarski and translating the new copies obtained, we have that $A^* \succcurlyeq S$, therefore we have $A^* \succcurlyeq S \succcurlyeq B^*$ and hence $A \succcurlyeq B$. ∎

3. The non-existence of paradoxes

We shall devote this section to showing that there is no analogue of the Banach-Tarski paradox in $\mathbb{R}$ or $\mathbb{R}^2$. This result was first shown by Banach in his paper [Ban23]. To do this we shall need to recall the Hahn-Banach theorem:

Theorem 3.1 (Hahn-Banach)**.** *Let V be a real vector space and U a subspace. Let ρ be a functional on V that satisfies $\rho(x+y) \leq \rho(x) + \rho(y)$(triangle-inequality) and $\rho(\lambda x) = \lambda\rho(x)$(positive-homogeneity) for all $x, y \in V$, $\lambda \in \mathbb{R}^+$. Let f be a linear functional defined on U such that $f(x) \leq \rho(x)$ for all $x \in U$. Then there exists a linear functional g defined on V that extends f, while satisfying $g(x) \leq \rho(x)$.*

We shall also need the notion of an *invariant mean*:

Given a group $\mathcal{G}$ with product $+$, let $\ell^\infty(\mathcal{G})$ denote the set of all bounded functionals on $\mathcal{G}$. Then an invariant mean is a linear functional on $\ell^\infty(\mathcal{G})$, satisfying the following two properties:

- $\inf_{x\in\mathcal{G}} f(x) \leq \mathcal{I}(f) \leq \sup_{x\in\mathcal{G}} f(x)$ for all $f \in \ell^\infty(\mathcal{G})$.
- $\mathcal{I}(s_g f) = \mathcal{I}(f)$ for all $g \in \mathcal{G}$ and $f \in \ell^\infty(\mathcal{G})$, where s_g is the operator from $\ell^\infty(\mathcal{G})$ to $\ell^\infty(\mathcal{G})$ such that $(s_g(f))(x) = f(x+g)$.

It turns out the following theorem holds:

Theorem 3.2. *If $(\mathcal{G}, +)$ is an abelian group then it has an invariant mean.*

Proof. Consider the functional ρ on $\ell^\infty(\mathcal{G})$ such that for $f \in \ell^\infty(\mathcal{G})$

$$\rho(f) = \inf\{\sup \frac{1}{n}\sum_{k=1}^{n} f(x+g_k) : n \in \mathbb{N}, (g_k)_{k=1}^{n} \in \mathcal{G}^n\}$$

Where the infimum is taken over finite-length sequences of elements of $\mathcal{G}$. Clearly we have positive homogeneity so let us verify the triangle inequality: given $f, g \in \mathcal{G}$ we can pick $\epsilon > 0$ such that there exist $(l_1, l_2, \ldots, l_n) \in \mathcal{G}^n$ and $(o_1, o_2, \ldots, o_m) \in \mathcal{G}^m$ such that

$$\sup \frac{1}{n}\sum_{k=1}^{n} f(x+l_k) < \rho(f) - \epsilon$$

and

$$\sup \frac{1}{m}\sum_{k=1}^{m} g(x+o_k) < \rho(g) - \epsilon$$

then we have that

$$\rho(f+g) \leq \sup\{\frac{1}{nm}\sum_{j=1}^{m}\sum_{k=1}^{n}(f+g)(l_k+o_j+x)\}$$

which implies that

$$\rho(f+g) \leq \frac{1}{m}\sum_{j=1}^{m}\sup\frac{1}{n}\sum_{k=1}^{n} f(l_k+o_j+x) + \frac{1}{n}\sum_{k=1}^{n}\sup\frac{1}{m}\sum_{j=1}^{m} f(l_k+o_j+x)$$

since $\mathcal{G}$ is abelian we have that $l_k + o_j = o_j + l_k$ therefore we have

$$\rho(f+g) \leq \frac{1}{m}\sum_{j=1}^{m}\sup\frac{1}{n}\sum_{k=1}^{n} f(l_k+o_j+x) + \frac{1}{n}\sum_{k=1}^{n}\sup\frac{1}{m}\sum_{j=1}^{m} f(o_j+l_k+x)$$

which in turn implies $\rho(f+g) \leq p(f)+p(g)-2\epsilon$, letting $\epsilon \to 0$ we get the triangle inequality.

Now, consider the subspace, U, of $\ell^\infty(\mathcal{G})$ that consists of all constant functionals. Let I be the functional on U that maps each $g \in U$ to its constant image. Clearly I is linear, and we have that $I \leq \rho$ on U, therefore by the Hahn-Banach theorem there exists a linear functional, $\mathcal{I}$, on $\ell^\infty(\mathcal{G})$ that extends I such that $\mathcal{I} \leq \rho$ on $\ell^\infty(\mathcal{G})$.
Let us now show that this $\mathcal{I}$ is an invariant mean:
Firstly, given two functions $f, g \in \ell^\infty(\mathcal{G})$ if $g(x) \leq f(x)$ for all $x \in \mathcal{G}$ then $\mathcal{I}(g-f) = \mathcal{I}(g) - \mathcal{I}(f) \leq \rho(g-f) \leq 0$ and hence $\mathcal{I}(g) \leq \mathcal{I}(f)$ which easily implies the first condition of being an invariant mean.
Secondly, for $g \in \mathcal{G}$ we have that $\rho(s_g(f)-f) \leq \sup \frac{1}{n}\sum_{k=1}^{n}(s_g(f)-f)(x+kg) \leq \frac{2}{n}\sup|f(x)|$, letting $n \to \infty$ we get that $\rho(s_g(f) - f(x)) = 0$. So, we have that $\mathcal{I}(s_g(x) - f(x)) \leq \rho(s_g(x) - f(x)) \leq 0$. Similarly one can show that $\mathcal{I}(f - s_g(f)) \leq \rho(f - s_g(f)) \leq 0$, which implies that $\mathcal{I}(s_g(f) - f) = 0$ therefore the second condition is satisfied. ■

An invariant mean shall allow us to construct a finitely additive measure, as seen by the following theorems:

Theorem 3.3. *There exists a finitely additive measure on all subsets $\mathbb{R}$, that is invariant with respect to translations. Hence, there is no analogue of Banach-Tarski in one dimension.*

Proof. Since the group $(\mathbb{R}, +)$ is abelian, there exists an invariant mean, $\mathcal{I}$, on it. Given a set $A \subseteq \mathbb{R}$, we can define the measure of it to be $\mathcal{I}(\chi_A)$, where χ_A is the indicator function of A, this gives us a finitely additive measure that is invariant with respect to translations. ∎

Theorem 3.4. *There exists a finitely additive measure on all subsets of* $\mathbb{R}^2$ *that is invariant with respect to the isometry group of the plane,* $\mathbb{E}^2$. *Hence, there is no analogue of Banach-Tarski in two dimensional space.*

Proof. Since the group of translations in the plane is abelian, by the same argument as before, there exists a finitely additive measure on all subsets of $\mathbb{R}^2$ that is invariant under translations, let us pick such a measure and call it m. Further since the circle group is abelian there exists an invariant mean, $\mathcal{I}$, on it. Now, for each $A \subseteq \mathbb{R}^2$ consider the function, F_A, from the circle to $\mathbb{R}$ that maps each angle, θ, to $m(f_\theta(A))$, where $f_\theta(A)$ is the set one obtains when the set A is rotated by an angle of θ. Now, $m'(A) = \mathcal{I}(F_A)$ gives us a finitely additive measure, m', that is invariant under both translations and rotations. This measure still may not be invariant under reflections. So consider the finitely additive measure $n(A) = m'(A) + m'(A^*)$ where A^* is the reflection of A about the line $y = x$. Clearly this is invariant under the reflection about the line $y = x$, rotations and translations and is therefore invariant with respect to $\mathbb{E}^2$. ∎

A group on which an invariant mean exists is called *amenable*[5]. It turns out that, by a difficult theorem of Tarski, a group is amenable if and only if it is not paradoxical. For a proof of this and several other results on amenable groups see theorem 12.11 in [WT16].

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[5]The term amenable is a pun, first used by M. Day. The notion of amenable groups was first considered by J. Von-Neumann.