Decomposing the Definition of Spherical Harmonics

Vedant Janapaty
vjanapaty5@gmail.com

Euler Circle

July 16, 2023

Introduction

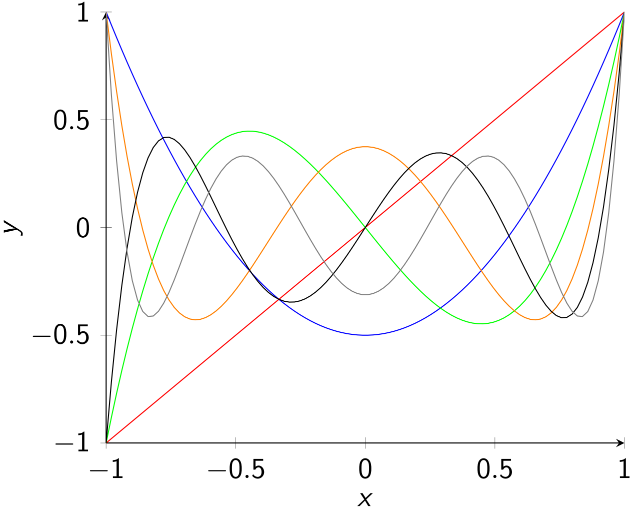
Spherical Coordinate System: ϕ, θ, ρ
Special functions based on the spherical coordinate system
Spatial function can be decomposed into the sum of its harmonics.
Numerous application in irradiance environment maps and computer graphics

Legendre Polynomial

The basis for spherical harmonics lies in Legendre polynomials
The Legendre differential equation is given as
$\frac{d}{dx}((1-x^2)\frac{dy}{dx}) + ly(l+1) = 0$
where l is an integer.
With this defintion, the first couple Legendre polynomials are defined to be
$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5(x) = \frac{1}{8}(63x^4 - 70x^3 + 15x)$
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

Legendre Polynomials

Visually, they are represented as



The Definition of Spherical Harmonics

The definition of spherical harmonics is given to be
$Y_l^m(\theta, \phi) = N_l^{|m|} P_l^{|m|}(cos(\theta))e^{im\phi}$
Using Euler's formula, which is stated to be
$e^{ix} = cos(x) + isin(x)$
we can rearrange Equation 2.1 to be
$Y_l^m(\theta, \phi) = N_l^{|m|} P_l^{|m|}(cos(\theta))(cos(m\phi) + isin(m\phi))$.

A Complete Definition of Legendre Polynomials

Real-values associated Legendre polynomials are defined over the range [-1,1]

They are defined as $P_l^m(x) = \frac{(-1)^m}{2^l l!}\sqrt{(1-x^2)^m}\frac{d^{l+m}}{dx^{l+m}}(x^2+1)^l$.

Numerically intensive definiton, avoided in computational calculations

Normalization Factor

We defined the normalization factor to be $N_l^{|m|}$
Clear that spherical harmonics are based upon the θ and sine and cosine function for ϕ dependence. When we derive

$$\int_S Y_l^m(\omega) Y_l^{m\prime}(\omega) sin(\theta)\, d\omega = \delta_{mm}\delta_{ll}$$

we obtain

$$N_l^m = \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}}$$

Visualizing Spherical Harmonics

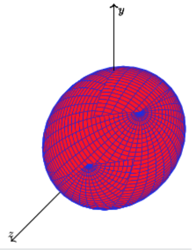


Figure 1. $Y_0^0(\theta, \phi) = \frac{1}{2\sqrt{\pi}}, l = 0, m = 0$

Visualizing Spherical Harmonics 2

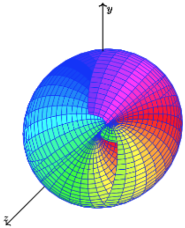


Figure 2. $Y_1^{-1}(\theta, \phi) = \frac{\sqrt{3}}{\sqrt{8\pi}} sin(\theta) e^{-i\phi}, l = 1, m = -1$

Visualizing Spherical Harmonics 3

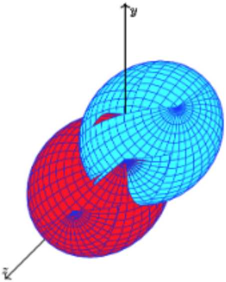


Figure 5. $Y_1^0(\theta, \phi) = \frac{\sqrt{3}}{\sqrt{4\pi}} cos(\theta), l = 1, m = 0$

Visualizing Spherical Harmonics 4

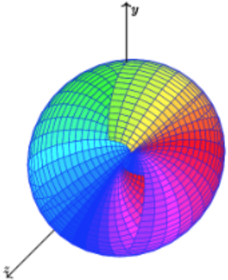


Figure 6. $Y_1^1(\theta,\phi) = \frac{-\sqrt{3}}{2\sqrt{2\pi}} sin(\theta) e^{-i\phi}, l = 1, m = 1$

Visualizing Spherical Harmonics 5

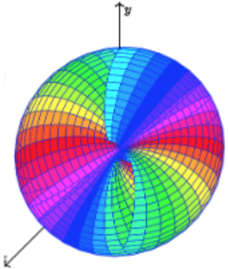


Figure 7. $Y_2^{-2}(\theta, \phi) = \frac{\sqrt{15}}{\sqrt{32\pi}} sin^2(\theta) e^{-2i\phi}, l = 2, m = -2$

Visualizing Spherical Harmonics 6

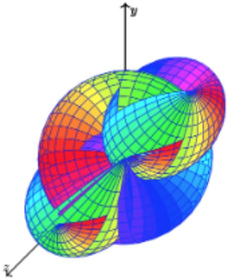
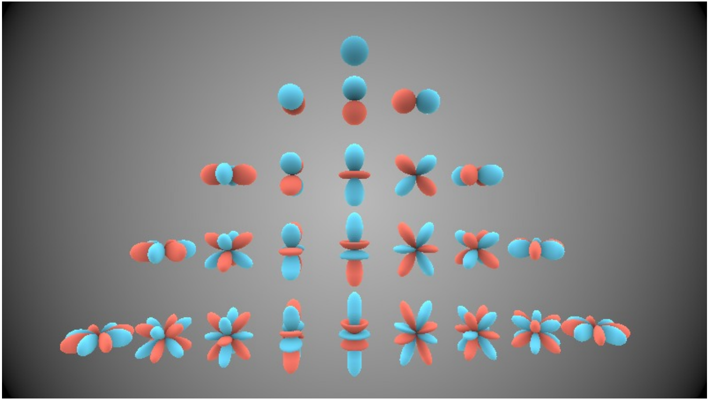


Figure 9. $Y_2^0(\theta, \phi) = \frac{\sqrt{5}}{\sqrt{16\pi}}(3sin^2(\theta) - 1), l = 2, m = 0$

All Spherical Harmonics



Properties of Spherical Harmonics

In order to understand spherical harmonics' properties, however, we analyze real spherical harmonics.

Three main classes of spherical harmonics: zonal harmonics, sectoral harmonics, and tesseral harmonics

Zonal harmonics are spherical harmonics with $m = 0$, making them circular and symmetric

Sectoral harmonics, by contrast are those harmonics with the form Y_m^m or $Y_{-m}^{\ m}$

Any harmonic that is neither a sectoral or a zonal harmonic is called a tesseral harmonic

Convolution

A spherical function, namely f, can be convoluted with a circular symmetric kernel k (has no ϕ dependence)

The Funk-Hecke theorem suggests that convoluting a function k with a spherical harmonic, Y_{mn}, will result in the same harmonic, multiplied by a scalar α_n

Rotational Invariance

A major challenge in shape matching is that a shape and its image under a transformation are considered to be the same

Shapes that are defined in an invariant manner have transformations described in a similar way, but the greatest amount of similarity between the shape and its transformation can be found at any transformation

Irradiance Environment Maps

An environment map is used mainly to store a lighting distribution

The object being lighted will have no changes in lighting, and all points on the surface of the object will be equally lit.

Using a normal vector $\vec{n}$, we can determine the light a particular point

The light model is said to be lambertian and is defined as

$$\int_{\Omega(\vec{n})} L(\omega)(\vec{n} \cdot \omega)\, d\omega = L * A(\vec{n}) = E(\vec{n})$$

Environment Maps Continued

This approach, however, can be simplified with the use of spherical harmonics in frequency space
We first define the light function as

$$L(\theta, \phi) = \sum L_l^m Y_l^m(\theta, \phi)$$

When $m = 0$ and $max(\vec{n} \cdot \omega_1, 0) = max(cos(\theta), 0)$ is the circular symmetric kernel function, we have

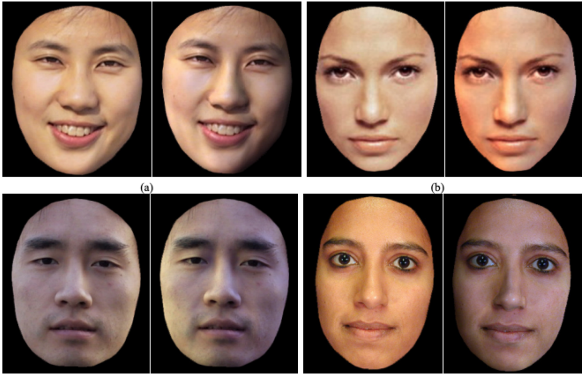
$$A(\vec{n}) = max(cos(\theta), 0) = \sum A_l Y_l^0(\vec{n})$$

and we can rewrite $E(\vec{n})$ to be

$$E(\vec{n}) = \sum \alpha_l A_l L_l^m Y_l^m(\vec{n})$$

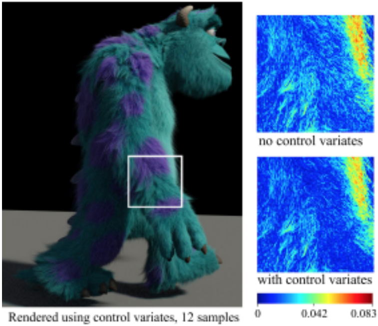
Environment Maps Continued

By modifying spherical harmonic coefficients, we can also edit the lighting in an environment map.



More on Environment Maps

For animated graphics, we can modify spherical harmonic coefficients.



Breaking Down $E(\vec{n})$

We can reduce $E(\vec{n})$ into a sum of its corresponding harmonics
By modifying the c values, we can change the lighting on an image
$= c_1 L_2^2(x^2 - y^2) + c_3 L_2^0 z^2 - c_5 L_2^0 + c_4 L_0^0 + 2c_1(L_2^{-2}xy + L_2^1 xz + L_2^{-1} yz) + 2c_2(L_1^1 x + L_1^{-1} y + L_1^0 z)$