Pell's Equation and Continued Fractions

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Introduction

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Generalized Pell's Equation: $x^2 - dy^2 = N$, where N is a nonzero integer.

Examples of Pell's Equation

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Example 2: $x^2 - 5y^2 = 1$

Example 3: $x^2 - 13y^2 = 1$

Introduction to Continued Fractions

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Continued Fraction: An expression of the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$.

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Simple Continued Fraction: All terms a_i are positive integers.

Introduction to Convergents in Continued Fractions

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Convergents of Simple Continued Fraction: If $[a_0; a_1, a_2, \ldots]$ is a simple continued fraction, then the convergents are given by the recurrence relations:

$$p_n = a_n p_{n-1} + p_{n-2},$$
$$q_n = a_n q_{n-1} + q_{n-2}.$$

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Actively used to find the fundamental solution and other solutions to Pell's Equation, especially when trial-and-error does not work.

Why: Imagine you are trying to find the fundamental solution to the Pell's Equation in which d equals to 109. Well, then, you need to try until when x equals to 158070671986249 and y equals to 15140424455100.

Solving Pell's Equation using Continued Fractions

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Theorem: If $\frac{p_n}{q_n}$ is a convergent of the continued fraction representation of $\sqrt{d}$, then (p_n, q_n) is a solution to the Pell's equation $x^2 - dy^2 = 1$.

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Algorithm:

1. Find the continued fraction representation of $\sqrt{d}$.
2. Compute the convergents $\frac{p_n}{q_n}$ until a solution to Pell's equation is found.

Fundamental Solutions of Pell's Equation

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Theorem: If (x_1, y_1) is a solution to Pell's equation, then (x_{n+1}, y_{n+1}) can be found using the recurrence relations:

$$x_{n+1} = x_0 x_n + d y_0 y_n,$$
$$y_{n+1} = x_0 y_n + y_0 x_n.$$

Solutions to Generalized Pell's Equation

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Theorem: If (u, v) is a solution to the Pell's equation $x^2 - dy^2 = 1$, then the general solution to the generalized Pell's equation $x^2 - dy^2 = N$ is given by:

$$x_n = ux_{n-1} + dvy_{n-1},$$
$$y_n = uy_{n-1} + vx_{n-1}.$$

Solution to Pell's Equation when $d = 61$

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Example: $x^2 - 61y^2 = 1$

Step 1: Find the continued fraction representation of $\sqrt{61}$.

Step 2: Compute the convergents $\frac{p_n}{q_n}$ until a solution to Pell's equation is found.

Solution: $(x, y) = (x_n, y_n)$ is a solution to Pell's equation.

Solution to Pell's Equation when $d = 61$

Example: $x^2 - 61y^2 = 1$

Step 1: Find the continued fraction representation of $\sqrt{61}$.

Step 2: Compute the convergents $\frac{p_n}{q_n}$ until a solution to Pell's equation is found.

Solution: $(x, y) = (x_n, y_n)$ is a solution to Pell's equation.

Solution: $(x, y) = (1766319049, 226153980)$ is the first solution to the Pell's equation when d=61.

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- Generalized Pell's equation $x^2 - dy^2 = N$ has solutions derived from the fundamental solutions.
- Continued fractions provide an efficient method to find solutions to Pell's equation and its generalizations.