Pell's Equation

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1 Introduction

Pell's Equation has been one of the most important and interesting equations in the Number Theory. Its history is as interesting as the equation itself. Although it is believed that Pell's Equation was first studied by John Pell in the seventeenth century, the history of Pell's Equation dates back to times of Indian mathematician Brahmagupta and Greek mathematician Pythagoras. Later, William Brouncker became the first European to solve the equation, and Leonhard Euler mistakenly attributed William Brouncker's solution to John Pell, which explains why the equation is named after John Pell . Pell's Equation has other forms, Generalized Pell's Equation and Negative Pell's Equation. Although the Generalized Pell's Equation is not used as commonly as the Pell's Equation, we still provided definition, solutions, and explanations for it. However, the Negative Pell's Equation is still subject to various different research and experiments. Thus, finding absolute information on it is still relatively hard. Owing to this, we did not talk about Negative Pell's Equation in this paper. Pell's Equation can be solved through many different ways, such as through convergents and fundamental solution via Continued Fractions or even through Quantum Algorithms. We can solve it by using trial-and-error method, or we can benefit from different theorems. In this paper, we mostly focused on solving it through convergents, fundamental solutions, and Continued Fractions. Continued Fraction is highly essential for mathematicians and understanding the mysteries behind some special irrational numbers. Furthermore, if we need to make calculations with the help of other irrational numbers such as $\sqrt{2}$, we take advantage of Continued Fractions without even realizing. Since we were aware of the fact that we needed to introduce the Pell's Equation, its othertypes, and its different solutions, we gave detailed and articulable explanations and definitions. Moreover, understanding the basics of the Continued Fraction is quite crucial to grasping the concept of the Pell's Equation. Due to this, we provided more than the basics of Continued Fraction. Thus, the section Continued Fraction can be used for other purposes, such as finding the best approximations to certain values, separately from understanding and solving the Pell's Equation. In this paper, all the solutions and explanations were solved by us and calculators, and they were explained, defined and written by us. The accuracy of the solutions and

2 CONTINUED FRACTIONS

Definition 2.1. (Olds, 1963) An expression of the following form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ddots}}}$$

is called as *continued fraction* where the $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are any real or complex numbers, and the number of terms is finite or infinite.

The purpose of the present section is to acquaint with the so-called regular continued fractions, that is, those of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

usually with the assumption that all the elements $a_1, a_2, a_3, \ldots,$ are positive integers.

Definition 2.2. (Rosen, 1992) An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is called as a *finite continued fraction* where $a_0, a_1, a_2, \ldots, a_n$ are real numbers with $a_1, a_2, \ldots, a_n$ positive. A finite continued fraction is denoted by $[a_0; a_1, a_2, \ldots, a_n]$ where the real numbers $a_1, a_2, \ldots, a_n$ are called the partial quotients of the continued fraction. The continued fraction is called *simple* if the real numbers $a_0, a_1, a_2, \ldots, a_n$ are all integers.

A finite continued fraction can also be written as $[a_0; a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_n]} = [a_0; [a_1, a_2, \ldots, a_n]]$ for $n > 0$.

Example 2.3. Express $[2;1,3,1,4]$ as a rational number.

$$[2;1,3,1,4] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}} = \frac{67}{24}$$

As can be seen, the value of any finite simple continued fraction is always a rational number and every rational number can be represented by a finite simple continued fraction (Rosen, 1992; Burton, 2010; Robbins, 1993).

Example 2.4. Express $\frac{67}{29}$ as a finite simple continued fraction.

By the Euclidean Algorithm, we have

$$67 = 2.29 + 9$$

$$29 = 3.9 + 2$$

$$9 = 4.2 + 1$$

$$2 = 2.1 + 0,$$

it follows that

$$\frac{67}{29} = 2 + \frac{9}{29} = 2 + \frac{1}{29/9} = 2 + \cfrac{1}{3 + \cfrac{2}{9}}$$

$$= 2 + \cfrac{1}{3 + \cfrac{1}{9/2}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}$$

$$= [2;3,4,2].$$

Since $2 = 1 + 1$, it can be written

$$\frac{67}{29}=2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1}}}}$$

Therefore, it can also be denoted as $[2;3,4,1,1]$.

This explains the following theorem.

Theorem 2.5. (Long, 1987) If $a_n > 1$, then $[a_0;a_1,a_2,\ldots,a_n]=[a_0;a_1,a_2,\ldots,a_n-1,1]$.

Definition 2.6. (Rosen, 1992) Let $A=[a_0;a_1,a_2,\ldots,a_n]$ where $\forall a_i\in\mathbb{R}$ with $a_1,a_2,\ldots,a_n$ positive. The continued fractions $C_k=[a_0;a_1,a_2,\ldots,a_k]$, where $k\in\mathbb{Z}$ with $0\le k\le n$, is defined as the *kth convergent of the continued fraction* $A=[a_0;a_1,a_2,\ldots,a_n]$ and it is denoted by C_k.

Theorem 2.7. (Stein, 2008) If real numbers p_k and q_k are defined as follows:

$$p_{-2}=0,\quad p_{-1}=1,\quad p_0=a_0,\quad p_1=a_1a_0+1,\quad \ldots\quad p_k=a_kp_{k-1}+p_{k-2}\quad \ldots,$$

$$q_{-2}=1,\quad q_{-1}=0,\quad q_0=1,\quad q_1=a_1,\quad \ldots\quad q_k=a_kq_{k-1}+q_{k-2},\quad \ldots,$$

then the *kth convergent* $C_k=[a_0;a_1,a_2,\ldots,a_k]$ is given by $C_k=\frac{p_k}{q_k}$ for $0\le k\le n$.

Theorem 2.8. (Burton, 2011)

a. The convergents with even subscripts form a strictly increasing sequence; that is, $C_0<C_2<C_4<\ldots$.

b. The convergents with odd subscripts form a strictly decreasing sequence; that is, $C_1>C_3>C_5>\ldots$.

c. Every convergent with an odd subscript is greater than every convergent with an even subscript.

In other words, this theorem briefly states that $C_0<C_2<C_4<\ldots<C_n<\ldots<C_5<C_3<C_1$.

Theorem 2.9. (Long, 1987) Let $\alpha=[a_0;a_1,a_2,\ldots,a_n]$ with $a_n>1$ so that α is the rational number $\frac{p_n}{q_n}$. Then, for $1\le i\le n$, we have that

$$\left|\alpha-\frac{p_i}{q_i}\right|<\left|\alpha-\frac{p_{i-1}}{q_{i-1}}\right|$$

and also

$$|\alpha q_i-p_i|<|\alpha q_{i-1}-p_{i-1}|.$$

Let's show what has been given so far on an example.

Example 2.10. Express $\frac{170}{39}$ as a finite simple continued fraction and compute the convergents for this simple continued fraction. Also, show that its continued fraction satisfies Theorem 2.5., Theorem 2.8. and Theorem 2.9.

By the Euclidean Algorithm, we have

$$170 = 4.39 + 14$$

$$39 = 2.14 + 11$$

$$14 = 1.11 + 3$$

$$11 = 3.3 + 2$$

$$3 = 1.2 + 1$$

$$2 = 2.1 + 0,$$

it follows that

$$\frac{170}{39} = 4 + \frac{14}{39} = 4 + \frac{1}{39/14} = 4 + \cfrac{1}{2 + \cfrac{11}{14}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{14/11}} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{3}{11}}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{11/3}}} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{2}{3}}}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3/2}}}} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

$$= [4; 2, 1, 3, 1, 2].$$

Since $2 = 1 + 1$, it can be written

$$\frac{170}{39}=4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}}}$$

Therefore, it can also be denoted as $[4;2,1,3,1,1,1]$. This satisfies Theorem 2.5.

The various convergents are

$C_0=[4]$ $\qquad C_0=\frac{p_0}{q_0}=\frac{4}{1}=4$

$C_1=[4;2]$ $\qquad C_1=\frac{p_1}{q_1}=\frac{9}{2}=4,5$

$C_2=[4;2,1]$ $\qquad C_2=\frac{p_2}{q_2}=\frac{13}{3}\approx 4,3333333333$

$C_3=[4;2,1,3]$ $\qquad C_3=\frac{p_3}{q_3}=\frac{48}{11}\approx 4,3636363636$

$C_4=[4;2,1,3,1]$ $\qquad C_4=\frac{p_4}{q_4}=\frac{61}{14}\approx 4,3571428571$

$C_5=[4;2,1,3,1,2]$ $\qquad C_5=\frac{p_5}{q_5}=\frac{170}{39}\approx 4,358974359$.

It is clear that $\frac{4}{1}<\frac{13}{3}<\frac{61}{14}<\frac{170}{39}<\frac{48}{11}<\frac{9}{2}$. Thus, $\frac{p_0}{q_0}<\frac{p_2}{q_2}<\frac{p_4}{q_4}<\frac{p_5}{q_5}<\frac{p_3}{q_3}<\frac{p_1}{q_1}$.
Therefore, $C_0<C_2<C_4<C_5<C_3<C_1$. This satisfies Theorem 2.8.

Let us check that if Theorem 2.9 is satisfied.

$$\left|\frac{170}{39}q_4-p_4\right|=\left|\frac{170}{39}14-61\right|=\left|\frac{1}{39}\right|\approx 0,0256410256$$

$$\left|\frac{170}{39}q_3-p_3\right|=\left|\frac{170}{39}11-48\right|=\left|\frac{-2}{39}\right|\approx 0,0512820513$$

From this we easily obtain $\left|\frac{170}{39}q_4-p_4\right|<\left|\frac{170}{39}q_3-p_3\right|$. It can be shown similarly for the others p_k and q_k $(1\leq k\leq 5)$.

Theorem 2.11. (Koshy, 2007) Let $C_k=\frac{p_k}{q_k}$ be the *kth convergent* of the simple continued fraction $[a_0;a_1,a_2,\ldots,a_n]$ where $1\leq k\leq n$. Then, $p_kq_{k-1}-q_kp_{k-1}=(-1)^{k-1}$ is valid.

The procedure of continued fraction can also be explained as follows.

Let $x \in \mathbb{R}$ and

$$x = \lfloor x \rfloor + \{x\} = a_0 + \{x\}$$

where $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \le \{x\} < 1$.

If $x \in \mathbb{Z}$, then this is the end of the algorithm.

If $x \notin \mathbb{Z}$, i.e. $\{x\} \neq 0$, then we write $x_1 = \frac{1}{\{x\}}$. Therefore

$$x = \lfloor x \rfloor + \frac{1}{x_1} \text{ with } x_1 > 1.$$

If $x_1 \in \mathbb{Z}$, then this is the end of the algorithm.

If $x_1 \notin \mathbb{Z}$, then we write $x_2 = \frac{1}{\{x_1\}}$. Therefore,

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{x_2}} \text{ with } x_2 > 1.$$

Set $a_0 = \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor$ for $i \ge 1$.

$$x = \lfloor x \rfloor + \cfrac{1}{\lfloor x_1 \rfloor + \cfrac{1}{\lfloor x_2 \rfloor + \cfrac{1}{\cdots}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

Consequently, $x = [a_0; a_1, a_2, \ldots]$. The algorithm finishes after finitely many steps if and only if x is rational.

Example 2.12. Let $x = \frac{24}{7}$. Then $x = 3 + \frac{3}{7}$ i.e. $a_0 = 3$ and $\{x\} = \frac{3}{7}$.

$x_1 = \frac{1}{\{x\}} = \frac{7}{3} = 2 + \frac{1}{3}$, so $a_1 = 2$ and $\{x_2\} = \frac{1}{3}$.

$x_3 = \frac{1}{\{x_2\}} = \frac{3}{1}$, so $a_2 = 3$ and $\{x_3\} = 0$.

Therefore, $x = \frac{24}{7} = [3; 2, 3]$.

Definition 2.13. (Burton, 1992) An *infinite continued fraction* is an expression of the following form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}$$

where $a_0, a_1, a_2, a_3, \ldots$ are real numbers with $a_1, a_2, \ldots, a_n$ positive and $a_0 \geq 0$ and it is denoted by $[a_0; a_1, a_2, \ldots, a_n, \ldots]$. If the real numbers $a_0, a_1, a_2, \ldots, a_n$ are all integers, then the continued fraction is called *simple*.

Theorem 2.14. (Rosen, 1992) Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers with $a_1, a_2, \ldots$ positive, and let $C_k = [a_0; a_1, a_2, \ldots, a_k]$. Then, the convergents C_k tend to a limit α, i.e. $\lim_{k\to\infty} C_k = \alpha$.

Definition 2.15. (Stein, 2008) *A periodic continued fraction* is a continued fraction of the form $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ such that $a_n = a_{n+t}$ for some fixed positive integer t and all sufficiently large n. Such a minimal t is called as the *period of the continued fraction*.

If the continued fraction contains no initial non-periodic terms, then it is called *purely periodic*.

Theorem 2.16. (Koshy, 2007) Let $\alpha = x_0$ be an irrational number. Define the sequence $\{a_k\}_{k=0}^{\infty}$ of integers a_k recursively as follows:

$$a_k = \lfloor x_k \rfloor, \qquad x_{k+1} = \frac{1}{x_k - a_k}$$

where $k \geq 0$. Then $\alpha = [a_0; a_1, a_2, \ldots]$.

Continued fraction expansion can also be found in the above form if α is an irrational number. Let's show this on an example.

Example 2.17. Express $\alpha = \sqrt{19}$ as an infinite simple continued fraction.

$$a_0 = \lfloor x_0 \rfloor = \lfloor \sqrt{19} \rfloor = 4, \qquad x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3}$$

$$a_1 = \lfloor x_1 \rfloor = 2, \qquad x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{19}+4}{3} - 2} = \frac{\sqrt{19}+2}{5}$$

$$a_2 = \lfloor x_2 \rfloor = 1, \qquad x_3 = \frac{1}{x_2 - a_2} = \frac{1}{\frac{\sqrt{19}+2}{5} - 1} = \frac{\sqrt{19}+3}{2}$$

$$a_3 = \lfloor x_3 \rfloor = 3, \qquad x_4 = \frac{1}{x_3 - a_3} = \frac{1}{\frac{\sqrt{19}+3}{2} - 3} = \frac{\sqrt{19}+3}{5}$$

$$a_4 = \lfloor x_4 \rfloor = 1, \qquad x_5 = \frac{1}{x_4 - a_4} = \frac{1}{\frac{\sqrt{19}+3}{5} - 1} = \frac{\sqrt{19}+2}{3}$$

$a_5 = \lfloor x_5 \rfloor = 2$, $\qquad x_6 = \dfrac{1}{x_5 - a_5} = \dfrac{1}{\dfrac{\sqrt{19}+2}{3} - 2} = \sqrt{19} + 4$

$a_6 = \lfloor x_6 \rfloor = 8$, $\qquad x_7 = \dfrac{1}{x_6 - a_6} = \dfrac{1}{\sqrt{19}+4-8} = \dfrac{\sqrt{19}+4}{3} = x_1$

As it can be seen that $x_7 = x_1$. So, the pattern continues. Thus,

$\sqrt{19} = [4; 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \ldots] = \left[4; \overline{2, 1, 3, 1, 2, 8}\right]$.

As can be seen, every irrational number can be represented by an infinite simple continued fraction (Koshy, 2007; Robbins, 1993).

Example 2.18. Express the purely periodic continued fraction $\alpha = \left[\overline{2;1}\right]$ in the form $a + b\sqrt{d}$, where $a, b \in \mathbb{Q}$ and d is a square-free integer greater than 1.

$$\left[\overline{2;1}\right] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}$$

Since $\alpha = \left[\overline{2;1}\right]$, it can be written

$$\alpha = 2 + \cfrac{1}{1 + \cfrac{1}{\alpha}} = \frac{3\alpha + 2}{\alpha + 1}.$$

That is, $\alpha^2 - 2\alpha - 2 = 0$, so $\alpha = 1 + \sqrt{3}$.

Every purely periodic continued fraction is an infinite continued fraction. As can be seen from the example, the value of an infinite continued fraction is an irrational number. This explains the following theorem.

Theorem 2.19. (Burton, 1992) The value of any infinite continued fraction is an irrational number.

Theorem 2.20. (Mollin, 2008) If $C_k = \dfrac{p_k}{q_k}$, for $k \in \mathbb{N}$, is the kth *convergent* of an irrational number α, then the following holds

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

For example, if $\sqrt{41} = \left[6; \overline{2, 2, 12}\right]$, it is obvious that $C_5 = \dfrac{p_5}{q_5} = \dfrac{2049}{320}$. So, $\left|\sqrt{41} - C_5\right| < \dfrac{1}{q_5^2}$, where C_5 is the $5th$ *convergent* in the infinite continued fraction representation of $\sqrt{41}$.

3 PELL'S EQUATION

3.1. Pell's Equation: Pell's Equation is a Diophantine equation. Pell's Equation are any equations where x and y are integers, and d is a positive integer but not a perfect square. That is,
$x^2 - dy^2 = 1$
The equation is extremely important in Number Theory since it comes with investigation and solution of numbers that are figurate in more than one way. Pell's Equation can give infinite number of solutions.

Definition 2.1. Diophantine equations are polynomial equations involving only sums, powers, and products. All the constants are integers, and the only solutions of interest are integers. That is,
$x^2 - y^2 = z^2$
where x, y, and z are integers.

Proposition 3.2. The reason why d cannot be a perfect square is that when d becomes a perfect square, we can only get one fundamental solution that is $(\pm 1, 0)$ for any positive integer d.
Proof. Let d be 4, a perfect square. Then, we have
$x^2 - 4y^2 = 1$
$x^2 - (2y)^2 = 1$
The only perfect squares that are 1 apart are —1— and 0. Thus, the only solution is $(\pm 1, 0)$.

Proposition 3.3. The reason why d cannot be a negative integer is that when d becomes a negative integer, we cannot get infinite number of solutions. *Proof.* Let d be -1, a negative integer. Then, we have
$x^2 - (-1y^2) = 1$
$x^2 + y^2 = 1$
Thus, the only solutions to this equation are $(\pm 1, 0)$ and $(0, \pm 1)$.
Proof. Let d be any negative integer that is smaller than -1. Thus, we will let d be -2. Then, we have
$x^2 - (2y)^2 = 1$
$x^2 + 2y^2 = 1$
Thus, the only solution to this equation is $(\pm 1, 0)$. That is, y cannot be greater or smaller than 0 because if it becomes any integer other than 0, then both x^2 and $-dy^2$ will become greater than 1, which cannot happen.

Definition 3.4. Fundamental solution refers to any solution which can solve one or more root causes. Thus, the root of the problem is used to construct theorems and problems based on them. That is, the fundamental solution of an equation is the smallest solution to that equation.

3.2. Generalized Pell's Equation: Generalized Pell's Equation is the equation where x and y are integers and d is any positive integer which is not a perfect square, and the solution is any integer except 1. That is,
$x^2 - dy^2 = n$
Generalized Pell's Equation uses its fundamental solution and its Pell's Equation form's fundamental solution to provide other solutions.

4 PELL'S EQUATION and CONTINUED FRACTIONS

Theorem 5: Suppose $d > 0$ is not a perfect square. Then expansion of $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_{n-1}, a_n, 2a_0}]$, where $a_{n+1-j} = a_j$ for $j = 1,2, \cdots, n$ (Olds 1963).

In other words, $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_2, a_1, 2a_0}]$

Proof

If $\sqrt{d} > 1 \Rightarrow -\sqrt{d} < -1$, then $\sqrt{d}$ is not reduced quadratic irrational.

Suppose

$$\sqrt{d} = [a_0; a_1, a_2, \cdots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{3}$$

Since $\sqrt{d} > 1$, is not a perfect square, then $a_0 + \sqrt{d} > 1$. We have, $0 < \sqrt{d} - a_0 < 1 \Rightarrow -1 < a_0 - \sqrt{d} < 0$ is a conjugate of $a_0 + \sqrt{d}$ that lies between -1 and 0. So, $a_0 + \sqrt{d}$ is a reduced quadratic irrational and it has a purely periodic continued fraction

We add a_0 in (3)

$$a_0 + \sqrt{d} = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Since the expansion of a reduced quadratic irrational is purely periodic. Then

$$a_0 + \sqrt{d} = 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}} = [\overline{2a_0; a_1, a_2, \ldots, a_n}]$$

$$\Rightarrow \sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + 2a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}} = \left[a_0; \overline{a_1, a_2, \ldots, a_n, 2a_0}\right]$$

By theorem 2, gives

$$\frac{-1}{a_0 - \sqrt{d}} = \frac{1}{\sqrt{d} - a_0} = \left[\overline{a_n, \ldots, a_1, 2a_0}\right]$$

Also, by subtracting a_0 from equation (3), we have,

$$\sqrt{d} - a_0 = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [\overline{0; a_1, a_2, \cdots, a_n, 2a_0}] \tag{4}$$

By theorem 3, it gives

$$\frac{1}{\sqrt{d} - a_0} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = \left[\overline{a_1, a_2, \cdots, a_n, 2a_0}\right] \tag{5}$$

Comparing equations (4) and (5), we have,
$a_n = a_1$, $a_{n-1} = a_2$, $\cdots$, $a_2 = a_{n-1}$, $a_1 = a_n$
Hence $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_2, a_1, 2a_0}]$.

Theorem 6: Let $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be an irreducible polynomial with integral coefficients and degree of $n \geq 3$. Let us consider the homogeneous polynomial

$$F(x,y) = y^n f\left(\frac{x}{y}\right)$$
$$= a_n x^n + a_{n-1} x^{n-1} y + \ldots + a_1 x y^{n-1} + a_0 y^n$$

Then the equation $F(x, y) = N$ has either no solution or only a finite number of solutions in integers (Thue 1909).

In the theorem 6 is in contrast when the degree of F is $n = 2$ (Dickson 1957).

For instance, if $F(x, y) = x^2 - dy^2$, where $d > 1$, is not a perfect square, then for non-zero integer N, the quadratic Diophantine equation of the form

$$x^2 - dy^2 = N \tag{6}$$

has either no integral solutions or infinitely many solutions, which is known as the generalized Pell's Equation (Dickson 1957) after John Pell, a Mathematician who studied in the *17th* century to find the integer solutions to equation (6).

Theorem 7: If $d > 1$, is not a perfect square integer, then $h_n^2 - dk_n^2 = (-1)^{n-1}$ q_{n+1} for every integer $n \geq -1$ (Kumundury & Romero 1998).

Theorem 7 gives us a solution to (6) for a given value of N. So, the following theorem 8 establishes the connection between the convergence of $\sqrt{d}$ and the solutions of equation (6) for $0 < N < \sqrt{d}$.

Theorem 8: Let $0 < N < \sqrt{d}$ and (P, Q) be a solution of the equation $x^2 - dy^2 = N$. Then $\frac{u}{v}$ is convergent in the expansion of $\sqrt{d}$ (Niven *et al.* 1991).

Proof

Since (u, v) is a solution of the equation (6), then

$N = u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d})$

Since $0 < N < \sqrt{d} \Rightarrow \sqrt{d} > N > 0$

Then, we have

$$0 < \frac{u}{v} - \sqrt{d} < \frac{N}{v\left(u + v\sqrt{d}\right)} < \frac{\sqrt{d}}{v\left(u + v\sqrt{d}\right)}$$

Since $v\sqrt{d} < u \Rightarrow 2v\sqrt{d} < u + v\sqrt{d}$

Then, we have

$$0 < \frac{u}{v} - \sqrt{d} < \frac{\sqrt{d}}{v\left(u + v\sqrt{d}\right)} < \frac{\sqrt{d}}{2v^2\sqrt{d}} = \frac{1}{2v^2}$$

$$\Rightarrow 0 < \frac{u}{v} - \sqrt{d} < \frac{1}{2v^2}$$

$$\Rightarrow \left|\frac{u}{v} - \sqrt{d}\right| < \frac{1}{2v^2}$$

It follows that $\frac{u}{v}$ is a convergent of $\sqrt{d}$.

Theorem 9: If $|N| < \sqrt{d}$, then the solutions of equation $x^2 - dy^2 = N$ are $x = u_n, y = v_n$, where $\frac{u_n}{v_n}$ is a convergent of $\sqrt{d}$.

Problem: Observe Pell's equation $x^2 - 7y^2 = 2$. Since $2 < \sqrt{7}$, we know that the solution (p_n, q_n) is a convergent $\frac{p_n}{q_n}$ of the continued fraction expansion of $\sqrt{7}$ .So, continued fraction expansion of $\sqrt{7} = [2; \overline{1,1,1,4}]$.

When $N = \pm 1$, the Diophantine equation (6) becomes

$$x^2 - dy^2 = \pm 1 \tag{7}$$

It is known as Pell's Equation, where $d > 0$, is not a perfect square. Pell's Equation uses a straightforward algebraic approach with a finite number of solutions when d>1 and a perfect square. Pell's Equation can be used to solve a variety of problems because it always has the trivial solution $(x, y) = (\pm 1, 0)$ and has an infinite number of solutions. The Indian mathematicians Brahmagumpta and Bhaskara developed techniques for resolving Pell's equations (Barbeau 2003). Pell's Equation can be resolved using the Chakravala method, which Brahmgupta first developed. These equations were used in the time of Pythagoras to approximate the square root of 2 (Pang 2011). So, Pell's Equation is also known as the classical Pell's Equation (Barbeau 2003, Niven *et al*. 1991) and Brahmagupta and Bhaskara were the first to study Pell's equation (Arya 1991).

The theory was developed by Lagrange, not Pell. Lagrange was the first to establish that there are infinitely many solutions to Pell's Equation, if d is a positive, not a perfect square (Legendre 1798). The Indian mathematician Baudhayana discovered in the fourth century that the equation $x^2 - 2y^2 = 1$ has a solution $(x, y) = (577,408)$, and he used the ratio $\frac{577}{408}$ to approximate $\sqrt{2}$.

But $\frac{577}{408} \approx 1.4142156$, while $\sqrt{2} = 1.4142135$ Archimedes estimated $\sqrt{3} \approx 1.7320508$ by $\frac{265}{153} \approx 1.7320261$ and $\frac{1351}{780} \approx 1.7320512$, then $\frac{x}{y}$ satisfy the equations $x^2 - 3y^2 = -2$ and $x^2 - 3y^2 = 1$. The smallest solution $(x, y) = (1151,120)$ to Pell's Equation $x^2 - 91y^2 = 1$, was investigated by

Brahmagupta in the seventh century. Similarly the least-positive solution $(x, y) = (1776319049, 2261590)$ to Pell's equation $x^2 - 61y^2 = 1$ was given by the Hindu mathematician Bhaskara in the twelve century.

Therefore, there are always infinitely many possible solutions to Pell's Equation (7). They can be found by continued fraction expansion of $\sqrt{d}$. The fundamental solution of equation (7) is usually the least positive solution. The following theorem 8 shows that if (x_1, y_1) is the fundamental solution to equation (7) then there are infinitely many solutions, and they are all generated from (x_1, y_1).

Another application of Pell's Equation is the approximation of square roots. Suppose that (x, y)satisfies Pell's equation. We cannot write , $\sqrt{d} = \frac{x}{y}$ where $x, y \in \mathbb{Z}$, $\sqrt{d}$ is irrational.

But, if $x^2 - dy^2 = 1 \Rightarrow \frac{x^2}{y^2} = d + \frac{1}{y^2} \approx d$. Therefore, Pell's solutions result in accurate rational approximations of $\sqrt{d}$. As a result, for large y, $\frac{x}{y}$ is a good approximation to $\sqrt{d}$. Therefore There are non-trivial solutions and infinitely many solutions to Pell's equation $x^2 - dy^2 = 1$. The fundamental solution, which is generated by Theorem 10, is at least one convergent of $\sqrt{d}$ and yields all solutions.

Theorem 10: Suppose $d > 0$ is not a perfect square. Then the continued fraction expansion of $\sqrt{d}=[a_0; \overline{a_1, a_2, \cdots, a_{r-1}, 2a_0}]$, where r is the length of period, then the fundamental solution (x_1, y_1) to Pell's equation (7) is given by the continued fraction expansion $\frac{x_1}{y_1}= [a_0; a_1, a_2, \cdots, a_{r-1}]$. Define $\frac{x_n}{y_n}= [a_0; a_1, a_2, \cdots, a_{nr-1}]$, then $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, for integer $n \geq 1, n \in \mathbb{Z}$ (Hoffstein *et al.* 2008).

Theorem 11: If (x_1, y_1) is the fundamental solution to Pell's equation (7), then n^{th} positive solution is (x_n, y_n), where x_n and y_n are given by $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, for integer $n > 1, n \in \mathbb{Z}$ (Waldschmidt 2016), which leads us to the following explicit form;

$$x_n = \frac{1}{2}\left\{\left(x_1 + y_1\sqrt{d}\right)^n + \left(x_1 - y_1\sqrt{d}\right)^n\right\},$$

$$y_n = \frac{1}{2\sqrt{d}}\left\{\left(x_1 + y_1\sqrt{d}\right)^n - \left(x_1 - y_1\sqrt{d}\right)^n\right\}$$

In addition, the solutions (x_n, y_n) satisfy the recurrence relations

$$x_{1+n} = 2x_1 x_n - x_{n-1}, \quad y_{1+n} = 2\, x_1 y_n - y_{n-1}$$

$$x_{1+n} = x_1 x_n + y_1 y_n d, \quad y_{1+n} = x_1 y_n + y_1 x_n$$

Theorem 12: Let $d > 0$, not a perfect square, and $\frac{p_n}{q_n}$ be the n^{th} convergent of $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_{r-1}, a_r}]$, where r is length of period.

All positive solutions of $x^2 - dy^2 = 1$are given by

$$(x, y) = \begin{cases} (p_{kr-1}, q_{kr-1}), k \in \mathbb{N}, if\ r\ is\ even \\ (p_{2kr-1}, q_{2kr-1}), k \in \mathbb{N}, if\ r\ is\ odd \end{cases}$$

All positive solutions of $x^2 - dy^2 = -1$ are given by

$$(x, y) = \begin{cases} (p_{kr-1}, q_{kr-1}), k \in \mathbb{N}, if\ r\ is\ odd \\ \quad no\ solution\ if\ r\ is\ even \end{cases}$$

Moreover, (p_{r-1}, q_{r-1}) is a fundamental solution of

$$\begin{cases} x^2 - dy^2 = 1, if\ r\ is\ even \\ x^2 - dy^2 = -1, if\ r\ is\ odd \end{cases}$$

and (p_{2r-1}, q_{2r-1}) is the fundamental solution of $x^2 - dy^2 = 1$ if r is odd (Niven *et al*. 1991).

We found a fundamental solution and used the fundamental solution to find other positive integral solutions to Pell's Equation.

Problem: Solve Pell's equation $x^2 - 41y^2 = 1$ using the method of continued fractions.

We begin by computing the continued fraction expansion of $\sqrt{41} = [6; \overline{2,2,12}]$. It has a length of period $r = 3$, which is odd. Therefore, negative Pell's Equation $x^2 - 41y^2 = -1$ has a solution. So, 3^{rd} convergent is

$C_2 = 6 + \cfrac{1}{2 + \cfrac{1}{2}} = 6 + \frac{2}{5} = \frac{32}{5}$. Thus, $(x, y) = (32,5)$ is a solution to the negative Pell's Equation.