# Fractal Dimension

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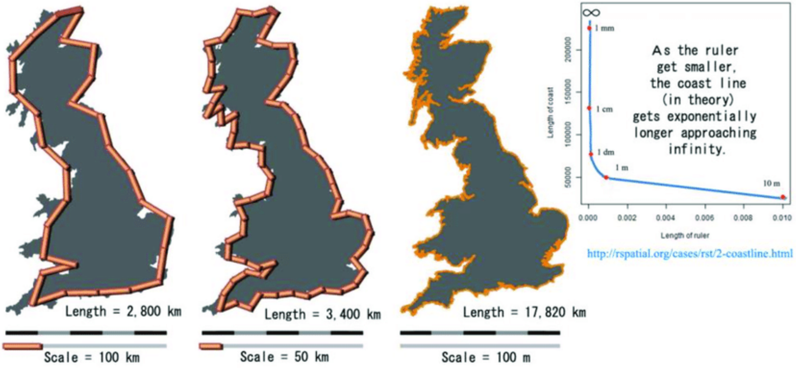
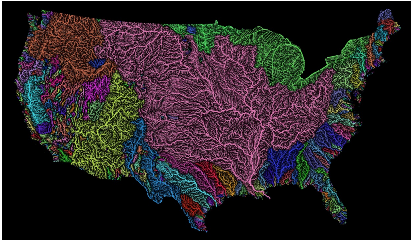
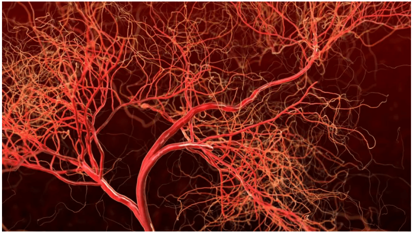
# The Coastline Paradox

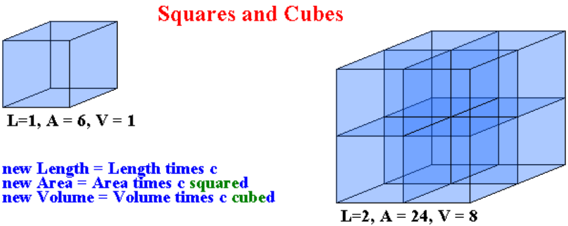


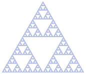
# Fractals in Nature - Infinite Detail

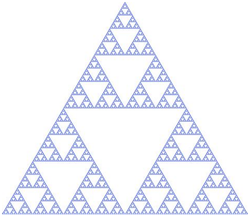




# Scaling





New Mass $=$ Mass times c to the $\frac{\log 3}{\log 2}$

# Fractal Dimension

The fractal dimension of a shape is found by scaling the fractal and seeing how its "mass" changes, with

$$\frac{\log \frac{N}{C}}{\log x}.$$

Here, $N$ is the mass after scaling, $C$ is the mass before, and $x$ is the scale factor.

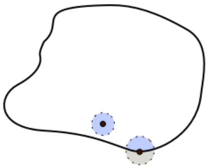
# Fractal

A fractal is a set with a fractal dimension strictly greater than its topological one.

## Definition

Topological Dimension A set $X$ has dimension 0 if for any point $P \in X$ there are arbitrary small neighborhoods with empty boundaries. $X$ has dimension $n$ if there are arbitrarily small neighborhoods around any $P$ with boundaries of dimension less or equal to $n - 1$.

## Example



# Examples

Figure: Koch Snowflake

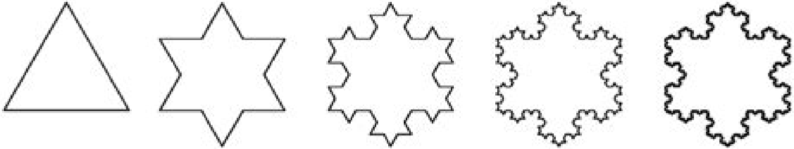


Figure: Menger Sponge

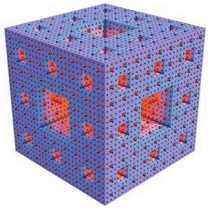
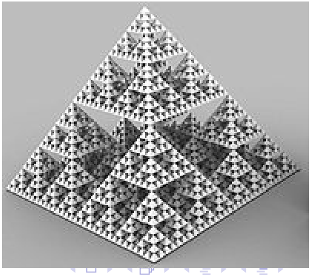


Figure: Pyramid Fractal



# Fractal Dimension of Self-Similar Fractals

A self similar fractal can be defined as a set $K = \bigcup r_i K$, where each $r_i K$ is a scaled down version of $K$ by some factor. If these sets are "kind of" disjoint, the fractal dimension is the solution $d$ to $\sum r_i^d = 1$.

# Hausdorff Measure

The $n$-Dimensional Hausdorff Measure of a set $X$ is defined as

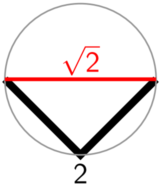
$$\mathscr{H}^n(X) = \lim_{\delta \to 0} \mathscr{H}^n_\delta(X),$$

where

$$\mathscr{H}^n_\delta(X) = \inf\{\sum \operatorname{diam}(E_i)^n \mid X \subseteq \bigcup E_i, \operatorname{diam}(E_i) \leq \delta\}.$$

This measure determines a size for fractal objects. It works by covering the set with a countable amount balls such that their combined volume is minimized. We then force the balls to get smaller.

We need to make the balls smaller as for "crumpled" objects, the size can be misrepresented.



# Hausdorff Dimension

The Hausdorff Dimension of a set $X$ is defined as follows:

$$\dim_{\mathscr{H}}(X) = \inf\{d > 0 \mid \mathscr{H}^n(X) = 0\}.$$

Alternatively, it can be defined as

$$\dim_{\mathscr{H}}(X) = \sup\{d > 0 \mid \mathscr{H}^n(X) = \infty\}.$$

This works because when $\delta$ gets smaller by a factor of $k$, $k^{-d}$ times more balls are needed to cover the $d$-dimensional set. However, the volume of each sphere will be multiplied by $k^n$. The total volume then is multiplied by $k^{n-d}$. If this value is anything other than 1, which is when $k = d$, the measure will veer towards 0 or $\infty$.

# Minkowski Dimension

The Minkowski Dimension is defined as two limits that must converge:
The Lower Minkowski Dimension is:

$$\underline{\dim_{\mathscr{M}}}(X) = \lim_{e \to 0} \inf \frac{\log N(e, A)}{\log \frac{1}{e}}.$$

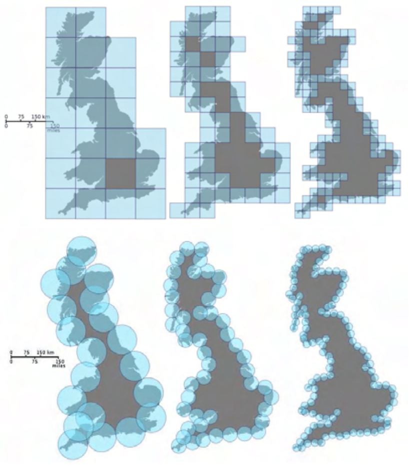
The Upper Minkowski Dimension is:

$$\overline{\dim_{\mathscr{M}}}(X) = \lim_{e \to 0} \sup \frac{\log N(e, A)}{\log \frac{1}{e}}.$$

The Minkowski Dimension involves covering a set with many congruent shapes, and seeing how the minimum amount changes as the size of the shapes decreases. Here, $N(e, A)$ is the number of shapes of diameter $e$ needed to cover object $A$.

# Visualization

Figure: Finding the Minkowski Dimension of the British Coastline using Box-Covering (Above) and Ball-Covering (Below). The coastline has a dimension of approximately 1.26.



# Example: Cantor Set

Figure: Cantor Set

If $3^{-n} < e \leq 3^{-(n-1)}$, at most $2^n$ balls will be needed, so $N(e, C) \leq 2^n$. Since every ball can intersect at most two segments, $N(e, C) \geq 2^{n-1}$. Therefore,

$$\overline{\dim}_{\mathscr{M}}(C) \leq \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3},$$

$$\underline{\dim}_{\mathscr{M}}(C) \geq \lim_{n\to\infty} \frac{\log 2^{n-1}}{\log 3^n} = \lim_{n\to\infty} \frac{n-1}{n} \times \frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.$$

Therefore, $\dim_{\mathscr{M}}(C) = \frac{\log 2}{\log 3}$.

# A Discrepancy

Though the Minkowski Dimension usually matches the Hausdorff Dimension, there are cases where they disagree. For example, consider the set $K$ containing points on the real line $\{\frac{1}{k} \mid k > 0\} \cup \{0\}$.
Since there are a countably infinite amount of elements, the Hausdorff Dimension is 0, but the Minkowski Dimension is $\frac{1}{2}$?

# A Discrepancy

For any $m$, $\frac{1}{m-1} - \frac{1}{m} = \frac{1}{(m)(m-1)} > \frac{1}{m^2}$. For any $1 > e > 0$, we can define an $n$ such that $\frac{1}{(n+1)^2} < e \leq \frac{1}{n^2}$, so $\frac{1}{\sqrt{e}} \geq n$. Covering $[1, \frac{1}{n}]$ needs $n$ balls of diameter $e$, since since the distance is greater than $e$. $[\frac{1}{n+1}, 0]$ can be covered by at most $n+1$ balls, since $e > \frac{1}{(n+1)^2}$. Therefore, we have

$$N(e, K) \leq n + n + 1 \leq 2e^{-\frac{1}{2}} + 1.$$

We also have that

$$N(e, K) \geq e^{-\frac{1}{2}},$$

since $e > \frac{1}{(n+1)^2}$, $e^{-\frac{1}{2}} < n + 1$ and $N(e, K) \geq n + 1$. Therefore, we have

$$e^{-\frac{1}{2}} \leq N(e, K) \leq 2e^{-\frac{1}{2}} + 1.$$

# A Discrepancy

We can now calculate the Minkowski Dimension.

$$\underline{\dim}_{\mathscr{M}}(K) = \lim_{e \to 0} \frac{\log e^{-\frac{1}{2}}}{\log \frac{1}{e}} = \frac{1}{2},$$

and

$$\overline{\dim}_{\mathscr{M}}(K) = \lim_{e \to 0} \frac{\log 2e^{-\frac{1}{2}} + 1}{\log \frac{1}{e}} = \frac{1}{2}.$$

Therefore, the Minkowski Dimension of K is $\frac{1}{2}$.
It seems to violate the property of the Hausdorff Dimension that if a set is a union of a finite or countably infinite amount of $X_i$, $\dim_{\mathscr{H}}(X) = \sup\{\dim_{\mathscr{H}}(X_i)\}$.