FRACTAL DIMENSION: QUANTIFYING THE INFINITE COMPLEXITY OF NATURE

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Abstract. We will start by defining the notion of fractals and fractal dimension, and finding it for some self-similar fractals. Then, we will explore the Hausdorff Dimension, where we will also define and prove properties about the Hausdorff Measure, which will be used to justify the Similarity Dimension on self-similar sets. We will then explore the Minkowski Dimension, and apply it on some non-self-similar sets, and also prove how it sometimes fails to agree with the Hausdorff Dimension.

1. Introduction

In many practical applications of mathematics, objects are assumed to be infinitely smooth. For example, calculus assumes that as a curve is zoomed in upon, it starts to approach a line, which implies the existence of a derivative. However, the world around us rarely holds up to these idealistic standards. Many things in nature are instead "infinitely rough," possessing detail at even infinitesimally small scales. This became apparent when Lewis Fry Richardson was researching the effect of the length of a border on the probability of war in 1950. He noticed that Portugal reported their border with Spain to be 987km, while Spain reported it as 1214km. This was the beginning of the coastline problem. The prevailing method of measuring a coastline or border was to place sticks of a certain length on a map such that both endpoints are on the coastline, and then measure the sum of their lengths. The measurement of the coastline would be this sum as the length of the sticks became smaller. However, with certain coastlines, the length did not converge as the measurement units get smaller, as they would on a smooth shape. Instead, they seemed to go to infinity, as seen in the measurement of the British Coastline.

Clearly, a better solution was needed to quantify such objects. Therefore, the fractal dimension was born, and these shapes that couldn't be measured were defined as "fractals." Fractal Dimension was first introduced by Felix Hausdorff in 1918, defining it as a measure of complexity for certain shapes. Benoit Mandelbrot, father of fractal theory, solidified this field in the 1960s and 70s. An object with an integer fractal dimension is a simple geometric shape. A dimension of 1, for example, can define a simple curve segment. A dimension of 1.1 can instead define a curve with small "bumps," as with the Gosper Island.

An object with dimension 1.8 could wind through space almost like a surface, like the 85° Koch Curve.

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Figure 1. The Coastline Paradox [Source]

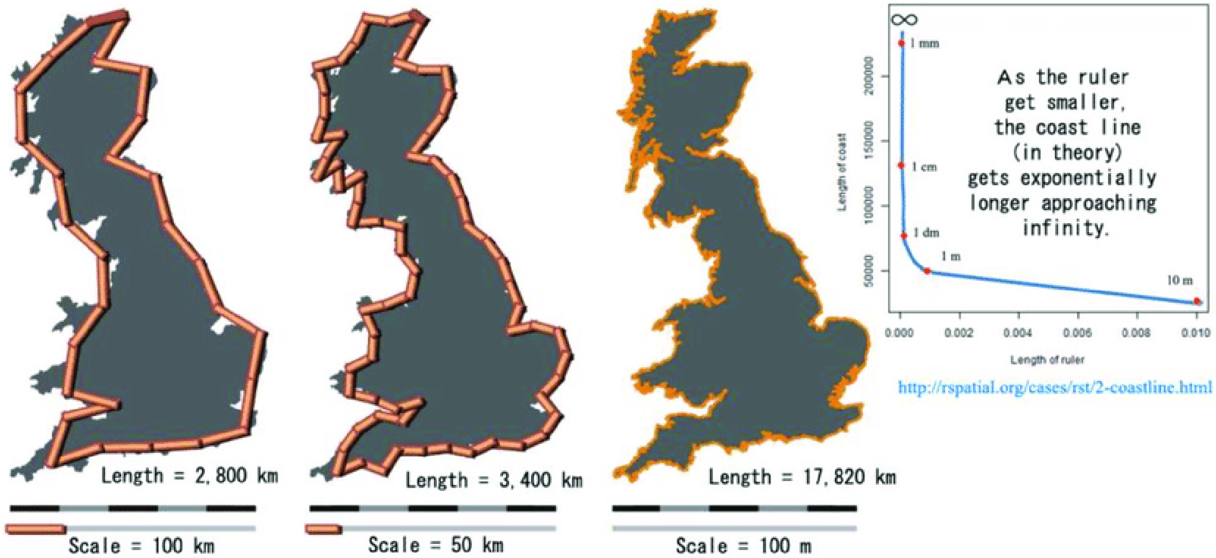


Figure 2. Gosper Island [Source]

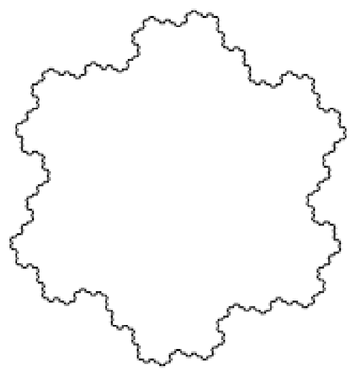
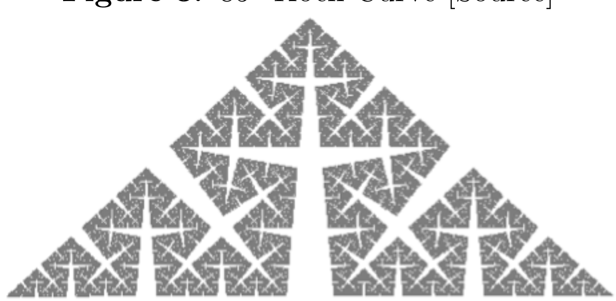


Figure 3. 85° Koch Curve [Source]



Since fractals are so prevalent, fractal dimension has a wide range of applications. It can be used in medical sciences, city planning, Brownian motion, and geography; wherever fractals are present.

2. Background

To begin our exploration, we need some background, especially from the field of measure theory.

Definition 2.1. Extended Real Numbers
The Extended Real Numbers are the set $\mathbb{R} \cup \{-\infty, \infty\}$.

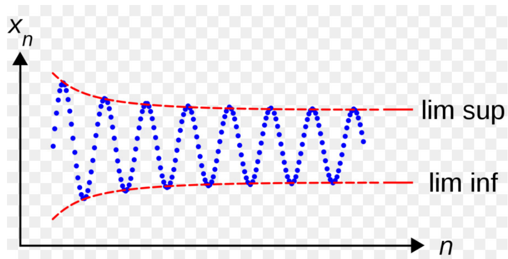
Definition 2.2. Countable
A set is countable if it is finite or there is a one-to-one mapping of the set to the set of natural numbers.

Definition 2.3. Infimum and Supremum
The infimum of $S \in X$ is the greatest element $x \in X$ such that for every $s \in S$, $s \geq x$. The supremum is the least element $x \in X$ such that $s \leq x$. They are often denoted as inf and sup, respectively.

More intuitively, the infimum is the lower bound of a set and the supremum is the upper bound. These are not the maxima and minima. For example, the infimum and supremum both exist and are 0 and 1 respectively in both $(0, 1)$ and $[0, 1]$, though the min and max don't exist for the second example.

Definition 2.4. Limit Infimum and Supremum
These are the limits of the lower and upper bounds, respectively. The former is denoted as $\liminf$, while the latter is $\limsup$.

Figure 4. [Source]



Definition 2.5. Covering
A covering of a set X is a set of sets E_i such that $X \subseteq \bigcup E_i$.

Definition 2.6. Metric Space
A metric space is an ordered pair (M, d) where M is a set and d, the metric of M, is a function such that $d : M \times M \to \mathbb{R}$, with the following axioms for all $x, y, z \in M$:

(1) $d(x, x) = 0$.
(2) If $x \neq y$, then $d(x, y) > 0$.
(3) $d(x, y) = d(y, x)$.
(4) $d(x, z) \leq d(x, y) + d(y, z)$.

The Euclidean Space $\mathbb{R}^n$ is of course a metric space, but another example could be the taxicab space, where distance is defined as the sum of the differences between each coordinate.

Definition 2.7. Separable Metric Space
A metric space (M, d) is separable if it contains a countable, dense, nonempty subset X. Here, "dense" means that for any element $x \in X$, there is an element of M that is arbitrarily close to it. In other words, $\inf\{d(m, x) \mid m \in M\} = 0$.

The Euclidean Space is obviously an example of this, as this applies for any subset. However, a metric space like $\mathbb{N}$ does not follow this, as every element is at least a distance of 1 from every other element.

Definition 2.8. Dilation Function
A dilation function (if it exists) on a metric space (M, d) by a factor r is one-to-one mapping function $f^r(a)$ that maps points such that $d(f^r(a), f^r(b)) = r \times d(a, b)$ for $a, b \in M$. A set X put through such a function can now be denoted as rX.

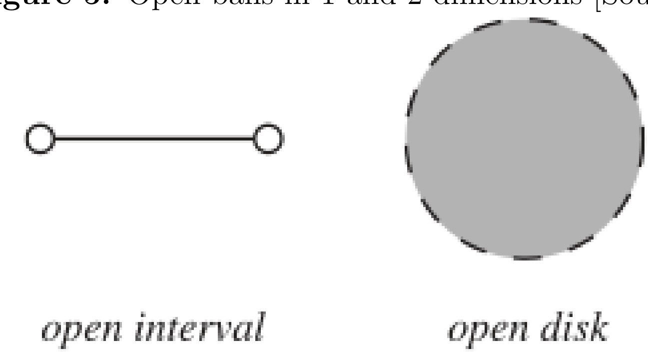
Definition 2.9. Diameter of a set
The diameter $\operatorname{diam}(X)$ of a set X in a metric space (M, d) is the largest distance between any two elements, defined as $\sup\{d(a, b) : a, b \in X\}$.

Definition 2.10. Distance Between Sets
In a metric space (M, d), the distance between two sets $A, B \subseteq M$ $d(A, B)$ is $\inf\{d(x, y) \mid x \in A, y \in B\}$. We can also use this definition to get the distance between a set A and an element x, which is the same as $d(A, \{x\})$.

Basically, this is the shortest distance between the two sets.

Definition 2.11. Open Ball
In a metric space (M, d), an open ball is defined as the set of all $x \in M \mid d(x, O) < r$ for center $O \in M$ and radius $r \in \mathbb{R} \mid r > 0$.

Figure 5. Open balls in 1 and 2 dimensions [Source]



Definition 2.12. Boundary Set
The boundary set of a set in a metric space is the set of all points such that any arbitrarily small open ball centered at one of those points contains elements from both within and outside of the set. We will denote the boundary set of X as X^B.

Definition 2.13. Open Set
An open set is a set X in metric space (M, d) such that for any $x \in X$, there exists a ε such that every point $y : d(x, y) < \varepsilon$ is in X.

Definition 2.14. Closed Set

A closed set is a set in metric space (M, d) defined as $M \setminus X$ for some open set X. Alternatively, it is a set that contains its boundary.

Definition 2.15. Compact Set

A set X is compact if for every covering of X by open sets, there exists a finite subset of the set of those open sets that also covers X.

Theorem 2.16. *All compact sets in metric space are closed*

If a set does not contain its entire boundary, which it has if it is finite, an infinite covering can be made such that the sets get smaller as they limit to the edge of an open boundary.

Definition 2.17. Outer Measure

An outer measure is a function that assigns a value in $[0, \infty]$ to all sets in a metric space with the following axioms:

(1) (Null Empty Set) $m(\emptyset) = 0$.
(2) (Subadditivity) For a set $A \subseteq X$ and a countable collection of subsets $B_1, B_2, \cdots \subseteq X$ such that $A \subseteq \bigcup B_i$, $m(A) \leq \sum m(B_i)$.

Definition 2.18. Metric Outer Measure

A metric outer measure is an outer measure over all sets metric space with the property of positively-separated additivity; if $A, B \subseteq X$ have $d(A, B) > 0$, $m(A \cup B) = m(A) + m(B)$.

Definition 2.19. Additive Measure An additive measure is a function over all sets in a metric space with the following properties:

(1) (Nonnegativity) For any set X, $m(X) \geq 0$.
(2) (Null Empty Set) $m(\emptyset) = 0$.
(3) (Countable Additivity) For a countable collection of disjoint sets $A_1, A_2, \cdots, m(\bigcup A_n) = \sum m(A_n)$.

This is normally known as a measure, but we will call it an "additive measure" for the sake of clarity.

3. Defining Fractals and the Fractal Dimension

Dimension is often defined as the amount of degrees of freedom in a space. Fractal dimension, instead, is defined with scaling. This dimension is derived from the power of a scale factor the volume is increased by. When a one-dimensional set is scaled up by 2, its volume scales by 2^1. For a two-dimensional set, it is scaled by 2^2, etc. This can be extended to non-integer dimensions, giving the definition below:

Definition 3.1. Fractal Dimension

The fractal dimension of a set X in metric space can be found by seeing how its "mass" changes when it is dilated by some factor. The fractal dimension is given by D such that

$$N = Cx^D,$$

where x is the scale factor, N is the mass after scaling, and C is the mass before scaling. Following from this, we can also get

$$D = \frac{\log(\frac{N}{C})}{\log(x)}.$$

The mass of a set can be defined differently depending on which measure of fractal dimension is used.

From this, the definition of a fractal can come:

Definition 3.2. Fractal
By Mandelbrot's definition, a fractal is a set in metric space with a fractal dimension strictly greater than its topological one.

We can also define a manifold as a set that is not a fractal.

Definition 3.3. Manifold
A manifold is a set in metric space with a fractal dimension equal to its topological dimension.

The topological dimension mentioned here is the dimension that a lot of mathematics, especially topology, is based on, and which is essentially derived from the idea of degrees of freedom. We can define topological dimension as follows:

Definition 3.4. Topological Dimension
In a separable metric space, a set X has dimension 0 if for any point $P \in X$ there are arbitrary small neighborhoods with boundaries containing no points in X. X has dimension n if there are arbitrarily small neighborhoods around any P with boundaries of dimension less or equal to $n-1$.

This definition unfortunately only works on separable metric spaces, which is sufficient in most cases. However, an additional definition, the Lebesgue Covering Dimension, is given as Definition 7.1 in the additional definitions section, though it is a lot less intuitive in its workings.

This definition of fractal dimension is not perfect for measuring fractals in nature, as their scaling is often not regular. Instead of smooth manifolds, fractal dimension approximates these shapes as self-similar shapes, with constant scaling factors.

Definition 3.5. Self-similar Set
A compact set X is self-similar if it can be defined as $\bigcup r_i X$ for some countable sequence of $r_i < 1$ and follows the Open Set Condition.

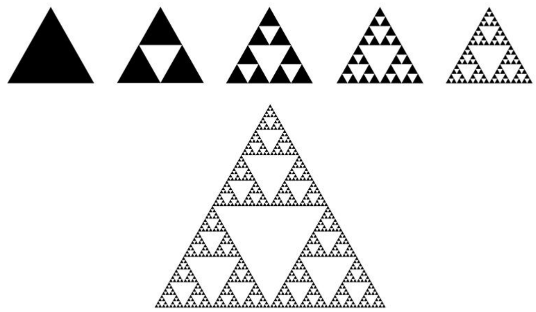
Definition 3.6. Open Set Condition
A set $X = \bigcup f_i^{r_i}(X)$ where the $f_i^{r_i}$ are the dilations by a factor of r_i satisfies the open set condition if there exists a non-empty open set V such that $\bigcup f_i^{r_i}(V) \subseteq V$ where each $f_i^{r_i}(V)$ is disjoint.

This ensures that the only overlaps a self-similar set has are the boundary sets of the components, as they will not be included within the open sets, unlike the rest of the set.

3.1. **Examples.**

Example. Sierpinski Triangle
The Sierpinski Triangle is formed by iteratively taking the midpoint of each side of a triangle and removing the points in the set within the triangle formed by the midpoints. Scaling the fractal by a scale factor of 2 produces 3 identical copies of the original triangle, resulting in a fractal dimension of $\frac{\log 3}{\log 2} \approx 1.585$.

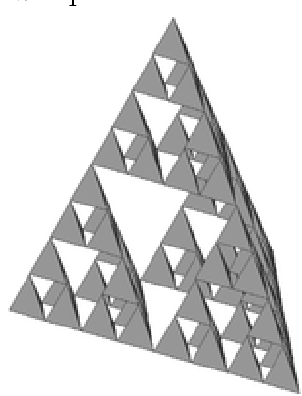
Figure 6. Sierpinski Triangle [Source]



We can calculate the topological dimension by noticing that for every point and every size of the neighborhood around it, we can place a circle around a smaller Sierpinski Triangle that the point is on, thus intersecting exactly three other points; the vertices of the triangle. Therefore, the dimension is at most 1. Since each neighborhood must intersect a triangle that it is smaller, the dimension is not 0, so the topological dimension is 1. This proves that the Sierpinski Triangle is a fractal, as $1 < 1.585$.

Example. Sierpinski Tetrahedron

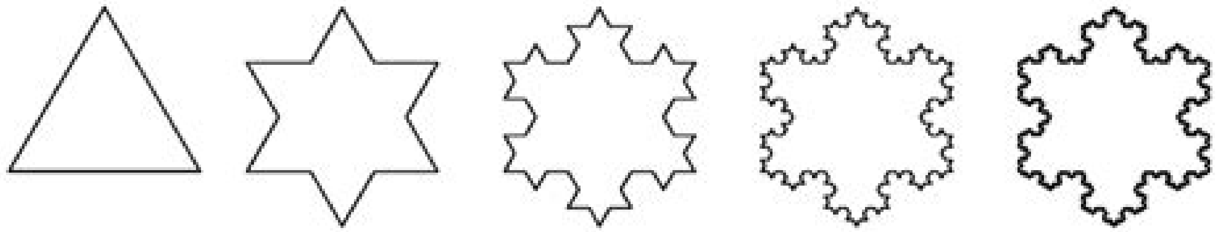
Figure 7. Sierpinski Tetrahedron [Source]



The Sierpinski Tetrahedron is formed by recursively replacing each tetrahedron with 4 tetrahedra. It has fractal dimension 2, as scaling by a factor of 2 results in a 4-fold increase in mass. However, even though it has an integer fractal dimension, it is still a fractal. We can see this by calculating the topological dimension, which is 1. The proof is the exact same as the one for the Sierpinski triangle, except that a sphere is used intersecting 4 points. Thus, the Sierpinski Tetrahedron has topological dimension 1, being a fractal with an integer fractal dimension as $1 < 2$.

Example. Koch Snowflake

Figure 8. Koch Snowflake [Source]



The Koch Snowflake is formed by recursively taking the center third of each segment and duplicating it, pushing both out. Therefore, a scale factor of 3 increases the mass by 4 times, giving a dimension of $\frac{\log 4}{\log 3} \approx 1.262$.

4. Hausdorff Dimension

The Hausdorff Dimension, developed by Felix Hausdorff, is one of the most popular ways to define and find fractal dimension, and essentially works by measuring a set in metric space with different dimensions, and finding the one that gives a useful result. The Hausdorff Dimension is defined as follows:

4.1. **Definition.**

Definition 4.1. Hausdorff Dimension
The Hausdorff Dimension of a set X is defined as follows:

$$\dim_{\mathscr{H}}(X) = \inf\{d > 0 \mid \mathscr{H}^n(X) = 0\}.$$

Alternatively, it can be defined as

$$\dim_{\mathscr{H}}(X) = \sup\{d > 0 \mid \mathscr{H}^n(X) = \infty\}.$$

Of course, before we understand the Hausdorff Dimension, we must first understand the measure.

4.2. **Hausdorff Measure.**

Definition 4.2. Hausdorff Measure
The n-Dimensional Hausdorff Measure of a set X is defined as

$$\mathscr{H}^n(X) = \lim_{\delta \to 0} \mathscr{H}^n_\delta(X),$$

where

$$\mathscr{H}^n_\delta(X) = \inf\{\sum \operatorname{diam}(E_i)^n \mid X \subseteq \bigcup E_i, \operatorname{diam}(E_i) \leq \delta\}.$$

for some countable sequence of E_i.

Definition 4.3. Hausdorff Measure δ-covering
A δ-covering of a set X is some set of E_i where $X \subseteq \bigcup E_i$ and $\operatorname{diam}(E_i) \leq \delta$.

The n-Dimensional Hausdorff Measure measures a set with n-dimensional units, getting its n-dimensional "mass." The Hausdorff Measure finds the smallest n-dimensional volume of a set of balls of various diameters that cover a set X as it limits the upper bound of the diameters to 0. The variable δ is necessary to force the balls to get smaller, as larger

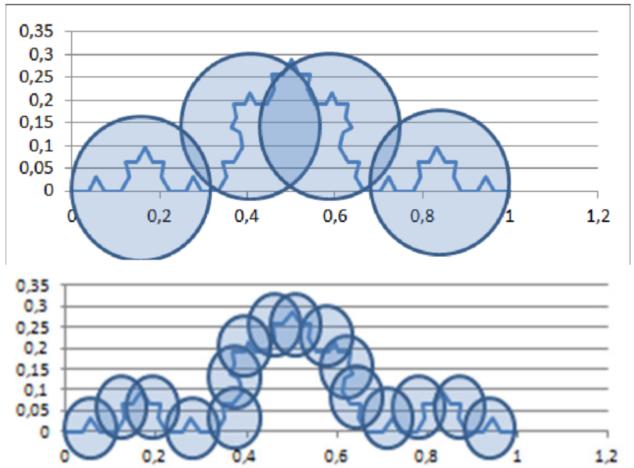
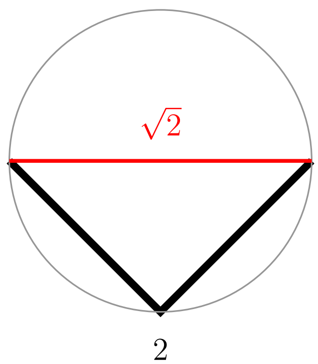


Figure 9. Hausdorff Measure on the Koch Snowflake. [Source 1] [Source 2]

measuring units can ignore the intricacies of a set (in bold) such as the one below, which is erroneously classified as much smaller than it actually is.



When δ is limited to 0, the units more closely follow the set. We therefore get the following theorem:

Theorem 4.4. *If* $\delta_1 < \delta_2, \mathscr{H}^n_{\delta_1}(X) \geq \mathscr{H}^n_{\delta_2}(X)$.

Proof. The δ_1-covering a valid δ_2-covering, as if all diameters are less than or equal to δ_1, they are less than or equal to δ_2. Therefore, if this is not true, then the δ_2-covering does not have infimal volume, as the δ_1-covering exists, invalidating our assumption and proving this inequality.

■

Theorem 4.5. *The Hausdorff Measure is a metric outer measure.*

Proof. The null empty set holds as it can be covered with 0 sets, giving an infimal value of 0 for all dimensions. Subadditivity holds as the concatenation of the coverings for some E_i is a valid covering, though not infimal, for $\bigcup E_i$, which is also a valid covering for $X \subseteq \bigcup E_i$, meaning that $\mathscr{H}^n(X) \leq \sum \mathscr{H}^n(E_i)$. To prove the third property, if $\delta < d(A, B)$, then the coverings for A and B in $A \cup B$ are separate systems as no covering set can have elements from both A and B, thus $\mathscr{H}^n(A) + \mathscr{H}^n(B) = \mathscr{H}^n(A \cup B)$ when $\delta \to 0$. ■

4.3. **Scaling.**

For the Hausdorff Measure to provide a useful measure that can be used in fractal dimension, it must follow the rules of fractal scaling.

Lemma 4.6. *For any set X in metric space and any $\delta > 0$, $\mathscr{H}^n_{k\delta}(kX) \leq k^n \times \mathscr{H}^n_\delta(X)$.*

Proof. Since dilating a metric space maps every point in X to a unique point in kX with a one-to-one correspondence, there is a one-to-one correspondence between the δ-coverings of X and the $k\delta$-coverings of kX. Therefore, for every covering of X with measure $\sum \operatorname{diam}(E_i)^n$, there is a covering with measure $\sum \operatorname{diam}(kE_i)^n = k^n \times \sum \operatorname{diam}(E_i)^n$. This means that $\mathscr{H}^n_{k\delta}(kX) \leq \sum \operatorname{diam}(kE_i)^n = k^n \times \mathscr{H}^n_\delta(X)$. ■

Lemma 4.7. *For any set X in metric space and any $\delta > 0$, $\mathscr{H}^n_{k\delta}(kX) = k^n \times \mathscr{H}^n_\delta(X)$.*

Proof. By Lemma 4.6, $\mathscr{H}^n_{k\delta}(kX) \leq k^n \times \mathscr{H}^n_\delta(X)$, and $\mathscr{H}^n_\delta(X) \leq (\frac{1}{k})^n \times \mathscr{H}^n_{k\delta}(kX)$, which is the same as $k^n \times \mathscr{H}^n_\delta(X) \leq \mathscr{H}^n_{k\delta}(kX)$, thus proving that $\mathscr{H}^n_{k\delta}(kX) = k^n \times \mathscr{H}^n_\delta(X)$. ■

Theorem 4.8. *$\mathscr{H}^n(kX) = k^n \times \mathscr{H}^n(X)$ for any X. In other words, Hausdorff Measure follows the rules of fractal scaling.*

Proof. By Lemma 4.7, $\mathscr{H}^n_{k\delta}(kX) = k^n \times \mathscr{H}^n_\delta(X)$. Limiting δ to 0 on both sides, we get $\mathscr{H}^n(kX) = k^n \times \mathscr{H}^n(X)$. ■

4.4. **Measurable Sets.**

The Hausdorff Measure has some additional properties on borel sets, as it is measurable, and an additive measure, on borel sets.

Definition 4.9. Borel Set

A Borel Set is defined as a set formed by the operations of countable union, intersection, and relative complement of open sets, alternatively closed sets.

Definition 4.10. Measurable

A set A is measurable by a measure m in metric space (M, d) if for any $B \subseteq M$,

$$m(B) = m(A \cap B) + m(B \setminus (B \cap A)).$$

Definition 4.11. σ-algebra

A set K that is a subset of the set of subsets of a set X is a σ-algebra on X if it has the following properties:

(1) $X \in K$.
(2) If $A \in K$, $X \setminus A$ is too.
(3) (Countable Union) For some countable amount of elements $A_1, A_2, \cdots A_n$, $\bigcup A_n$ is in K.

Some other properties can be implied from the previous ones:

(4) (Countable Intersection) If $A_1, A_2, \cdots A_n \in K$, $\bigcap A_n \in K$.
This works because $\bigcap A_n = X \setminus \bigcup (X \setminus A_n)$.
(5) (Relative Complement) If $A, B \in K$, $A \setminus B \in K$.
This works because $A \setminus B = A \cap (X \setminus B)$.

Theorem 4.12. *The Set of Borel Sets is a σ-algebra, called the Borel σ-algebra.*

Theorem 4.13. *Caratheodory's Theorem*

The set of all measurable sets K for a metric outer measure m in (M, d) is a σ-algebra. In addition, m is an additive measure over K.

Proof. Clearly, $\emptyset, M \in K$. If $A \in K$, then for $B \subset M$, $m(B) = m(A \cap B) + m(B \setminus (B \cap A) = B \cap (M \setminus A))$, so $M \setminus A \in K$. Now, suppose we have some $A_j \in K$, and $A = A_1 \cup A_2$. We have

$$m(Y) \leq m(Y \cap (A_1 \cup A_2)) + m(Y \cap (M \setminus (A_1 \cup A_2))).$$

We can redefine A as a disjoint union $A_1 \cup (A_2 \cap (M \setminus A_1))$. Therefore, the previous sum is less or equal to

$$m(Y \cap A_1) + m(Y \cap (A_1 \cap (M \setminus A_1))) + m(Y \cap (M \setminus A_1) \cap (M \setminus A_2)),$$

which, since $M \setminus A_2$ is measurable, is equal to $m(Y \cap A_1) + m(Y \cap (M \setminus A_1)) = m(Y)$. Therefore, K is closed under finite unions. Before we prove this for countable unions, we must prove additivity. If measurable sets $A, B \in K$ are disjoint, we have

$$m(A \cup B) = m((A \cup B) \cap A) + m((A \cup B) \cap (M \setminus A)) = m(A) + m(B).$$

We can extend this to any finite amount of sets by induction. If we have a countably infinite amount of disjoint $A_i \in K$, for any n, we have

$$\sum_{i=1}^{n} m(A_i) = m(\bigcup_{i=1}^{n} m(A_i)) \leq m(\bigcup_{i=1}^{\infty} m(A_i)).$$

Thus, as $n \to \infty$,

$$\sum_{i=1}^{\infty} m(A_i) \leq m(\bigcup_{i=1}^{\infty} m(A_i)).$$

Since subadditivity proves the opposite inequality, $\sum_{i=1}^{\infty} m(A_i) = m(\bigcup_{i=1}^{\infty} m(A_i))$. Thus, m is an additive measure as it is additive for all disjoint countable unions. Now, we can prove that the countably infinite union is measurable. For $A_i \in K$, let $B_n \in K = \bigcup_{i \leq n} A_i$. Lets define $A_i' = B_n \setminus B_{n-1}$, which is in K as it is equal to $M \setminus ((M \setminus B_n) \cup (M \setminus B_{n-1}))$. This means that all A_i' are disjoint, and $\bigcup_{i \leq n} A_i = \bigcup_{i \leq n} A_i'$. With $A = \bigcup^{\infty} A_i$, we have for any $S \in M$,

$$m(S) = m(S \cap B_n) + m(S \cap (M \setminus B_n)) \geq \sum_{i=1}^{n} m(S \cap A_i') + m(S \cap (M \setminus A)).$$

When $n \to \infty$,

$$m(S) \geq m(S \cap (M \setminus A)) + \sum_{i=1}^{\infty} m(S \cap A_i') = m(S \cap (M \setminus A)) + m(\bigcup S \cap A_i'),$$

which is equal to

$$m(S \cap (M \setminus A)) + m(S \cap A).$$

This is opposite to the subadditive relation, so this value and $m(S)$ are equal, proving that A is measurable, proving that countable unions are in K. This completes the proof that K is a σ-algebra, and that m is a measure over K. [Tay06]

■

Lemma 4.14. *All closed sets are measurable by a metric outer measure m.*

Proof. Let A be a closed set in metric space (M, d), and B be another set in the metric space. Since m is an outer measure, we have

$$m(A) \leq m(A \cap B) + m(A \setminus (A \cap B)).$$

If $m(A) = \infty$, this clearly holds for all A. Otherwise, for each n, let

$$B_n = \{x \in (B \setminus (B \cap A)) : d(x, F) \geq n^{-1}\},$$

which has that $d(B_n, A) \geq n^{-1}$. Since $d(B_n, B \cap A) \geq d(B_n, A)$,

$$m(B_n \cup (A \cap B)) = m(B_n) + m(A \cap B).$$

Since A is closed, for any $x \in B \setminus (B \cap A)$, $d(x, F) > 0$, so

$$B \setminus (B \cap A) = \bigcup_{n=1}^{\infty} B_n.$$

Therefore,

$$B = (B \cap A) \cup \bigcup_{n=1}^{\infty} B_n.$$

We also have

$$m(B) \geq m((A \cap B) \cup B_n) = m(A \cap B) + m(B_n).$$

To prove that $m(B) \geq m(A \cap B) + m(B \setminus (A \cap B))$, we need

$$m(B \setminus (A \cap B)) = \lim_{n \to \infty} m(B_n).$$

Lets have $D_n = B_{n+1} \setminus (B_{n+1} \cap B_n)$. For $x \in D_{n+1}$ and $y \in M$ satisfying $d(x, y) < \frac{1}{(n+1)n}$, we have

$$d(y, A) \leq d(x, y) + d(x, A) < \frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}.$$

This implies that $y \notin E_n$, meaning that

$$d(D_{n+1}, B_n) \geq \frac{1}{(n+1)n}.$$

From this, we have

$$m(B_{2n+1}) = m(D_{2n} \cup B_{2n}) \geq m(D_{2n} \cup B_{2n-1}) = m(D_{2n}) + m(B_{2n-1}).$$

If we continue to substitute for B_{2n-1}, we get that

$$m(B_{2n+1}) \geq m(D_{2n}) + m(D_{2n-2}) + \cdots + m(D_2) + m(B_1) \geq \sum_{j=1}^{n} m(D_{2j}).$$

We also have

$$m(B_{2n}) = m(D_{2n-1} \cup B_{2n-1}) \geq m(D_{2n-1} \cup B_{2n-2}) = m(D_{2n-1}) + m(B_{2n-2}).$$

Again, we substitute and get

$$m(B_{2n}) \geq m(D_{2n-1}) + m(D_{2n-3}) + \cdots + m(D_1) + m(B_0) = \sum_{j=1}^{n} m(D_{2j-1)}.$$

Since $B_n \subset B$, $m(B_n) \leq m(B)$. Thus, both of these series we found converges to a value less or equal to $m(B)$, so the series $\sum m(D_j)$ converges to a value less or equal to $2m(B)$. We know that

$$m(B \setminus (B \cap A)) = m(B_n \cup \sum_{j=n}^{\infty} D_j) \leq m(B_n) + m(\sum_{j=n}^{\infty} D_j)$$

For any n. As $n \to \infty$, $m(\sum_{j=n}^{\infty} D_j)$ converges to 0, as the sum converges. Therefore, since $B_n \subseteq B \setminus (B \cap A)$,

$$m(B \setminus (B \cap A)) \leq \lim_{n\to\infty} m(B_n) \leq m(B \setminus (B \cap A)).$$

Therefore,

$$m(B) = m(B \cap A) + m(B \setminus (B \cap A)).$$

[Bel14]

■

Theorem 4.15. *All borel sets are measurable by a metric outer measure m.*

Proof. By Caratheodory's Theorem, the set of all measurable sets is a σ-algebra. For a set to be part of the borel set, it must be formed by closed sets through the operations of countable union, intersection, and relative complement. Since sets formed by these operations on some base sets are part of the σ-algebra the base sets are in, the sets in the borel set are part of this measurable σ-algebra too. ■

Claim 4.16. *The Hausdorff Measure is an additive measure and is measurable on the borel σ-algebra.*

Proof. Since the Hausdorff Measure is a metric outer measure, this follows from Caratheodory's Theorem and Theorem 4.15. ■

4.5. **Lebesgue Measure.**

The Hausdorff Measure can also be thought of as an extension of the Lebesgue Measure, which is defined in euclidean spaces:

Definition 4.17. Lebesgue Measure

The n-dimensional Lebesgue Measure is a function on a set $X \subseteq \mathbb{R}^n$ that returns its content as a non-negative real number, which is defined as follows:

$$\mathscr{L}^n(X) = \inf\{\sum \mathrm{vol}(C_k) : (C_k)_{k\in\mathbb{N}} \text{ is a sequence of } n\text{-dimensional rectangular cuboids} \\ \text{with } X \in \bigcup C_k\}.$$

An n-dimensional rectangular cuboid is a product of n intervals.

Explained intuitively, we cover the set with rectangular cuboids such that the sum of these volumes is the least. The minimum such volume is the volume of the set, as the union of these cuboids must cover this set.

Instead of spheres, we cover with cuboids, and a max diameter is not needed since the dimension of the measure is the dimension of the space the set is defined in. Otherwise, the Lebesgue Measure and Hausdorff Measure are equal if the Hausdorff Measure is multiplied by some fixed constant for each dimension, since the volume of the covering balls are off by a constant from their euclidean volumes. This shows how the Hausdorff Measure correctly measures manifolds.

4.6. **Dimension (Continued).**

Now that we know how the Hausdorff Measure works, we can understand the Hausdorff Dimension. A basic argument for why it works would be that when δ decreases by a factor of k, $(\frac{1}{k})^d$ times as many units with diameters scaled by k are needed to cover a set X with dimension d. For an n-dimensional Hausdorff Measure, the volumes of the units scale by k^n, making the total volume scale by $\frac{k^n}{k^d}$. If $d > n$, then this value will be greater than 1, making the measure go to infinity as δ goes to 0. Alternatively, when $n > d$, the value is less than 1, making the measure go to 0 as δ goes to 0. The only time that the measure will return a correct value is when $d = n$, as the value will be 1. Basically, the Hausdorff Dimension finds the measure that scales at the same rate as the fractal.

Theorem 4.18. *If set X is a union of a finite or countably infinite amount of X_i,*

$$\dim_{\mathscr{H}}(X) = \sup\{\dim_{\mathscr{H}}(X_i)\}.$$

Proof. If we take an $n > \sup\dim_{\mathscr{H}}(X_i)$, $\mathscr{H}^n(X) = 0$. This is because for every X_i, $\mathscr{H}^n(X_i) = 0$ as $\dim_{\mathscr{H}}(X_i) < n$. The subadditivity property implies that

$$\mathscr{H}^n(X) \leq \sum \mathscr{H}^n(X_i) = 0.$$

Since there is always an ε such that $\sup\dim_{\mathscr{H}}(X_i) < \varepsilon < n$, The n-dimensional measure of X is not the infimum of all measures that result in 0 as $\mathscr{H}^\varepsilon(X) = 0$, meaning that

$$\dim_{\mathscr{H}}(X) \leq \sup\dim_{\mathscr{H}}(X_i).$$

If we instead take an $n < \sup\dim_{\mathscr{H}}(X_i)$, we can find some X_i K where $\dim_{\mathscr{H}}(K) > n$. Since $K \subseteq X$,

$$\mathscr{H}^n(X) \geq \mathscr{H}^n(K) = \infty.$$

Again, we find a $n < \varepsilon < \sup\dim_{\mathscr{H}}(X_i)$, which proves that The n-dimensional measure is not the supremum of all measures that result in ∞ as $\mathscr{H}^\varepsilon(X) = \infty$, proving that

$$\dim_{\mathscr{H}}(X) \geq \sup\dim_{\mathscr{H}}(X_i).$$

Therefore,

$$\sup\dim_{\mathscr{H}}(X_i) \leq \dim_{\mathscr{H}}(X) \leq \sup\dim_{\mathscr{H}}(X_i) \to \dim_{\mathscr{H}}(X) = \sup\dim_{\mathscr{H}}(X_i).$$

■

5. Similarity Dimension

The similarity dimension is a method used to find the dimension of self-similar sets, which generalizes our method for the previous example fractals. We can prove that it is equal to the Hausdorff Dimension.

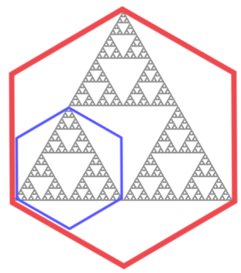
Definition 5.1. Similarity Dimension

If a set X is self-similar, the similarity dimension of the set is the solution d to $\sum r_i^d = 1$.

An informal explanation of this could be that since the overlaps are proven to be "insignificant" by the open set condition, the set is simply a sum of the masses of the smaller components. By the rules of fractal scaling, if the set has dimension d, each component has a mass r_i^d times the mass of the full set. These must add to the full set's mass, so this equality must be true.

We can see how this works with the Sierpinski Triangle, as it follows the open set condition with the following covering:

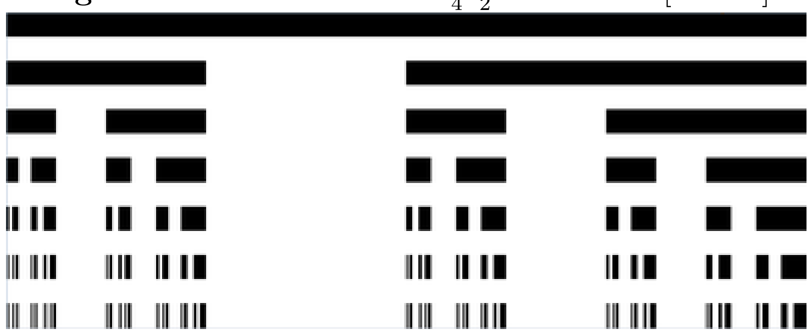
Figure 10. Open Set Covering of the Sierpinski Triangle [Source]



We can find the fractal dimension to be the solution of $3 \times \frac{1}{2}^d = 1$, which is $\frac{\log 3}{\log 2}$.

We can also apply this to self-similar sets with scaled sets that are not all the same size. For example, the following set, defined as $X = \frac{1}{2}X \cup \frac{1}{4}X$ has dimension 0.6942, as it is the solution to $(\frac{1}{2})^d + (\frac{1}{4})^d = 1$.

Figure 11. Iterations of the $\frac{1}{4}$-$\frac{1}{2}$ Cantor Set [Source]



To prove that the Similarity Dimension is equal, we first need the following theorems:

Lemma 5.2. *All compact sets are borel sets.*

Proof. All compact sets are closed, and all closed sets are borel sets, as they can be formed by the complement of an open set. ■

Theorem 5.3. *A self-similar set $X = \bigcup (r_i X = f_i^{r_i}(X))$ has $\mathscr{H}^k(f_i^{r_i}(X) \cap f_j^{r_j}(X)) = 0$ for $i \neq j$, where $k = \dim_{\mathscr{H}}(X)$.*

An informal reason for this is that the overlaps are not "significant," thus making their measures 0.

Theorem 5.4. *The Hausdorff Dimension is equal to the similarity dimension if $0 < \mathscr{H}^k(X) < \infty$ for a self-similar set $X = \bigcup r_i X$.*

Proof. Given that $0 < \mathscr{H}^k(X) < \infty$, the Hausdorff Dimension of X is k. Since X is compact, it is a borel set, which also applies to each $r_i X$. For each m, the intersection of $r_m X$ with all other $r_j X$, $\bigcup r_m X \cap r_j X \mid j \neq m \subseteq \bigcup r_i X \cap r_j X \mid j \neq i$, which we will denote as $r_m X'$, is also a borel set, as it is the union of intersections of other borel sets. It also has measure 0, as $\mathscr{H}^k(r_m X') \leq \mathscr{H}^k(\bigcup r_i X \cap r_j X) \leq \sum \mathscr{H}^k(r_i X \cap r_j X) = 0$. $r_m X \setminus r_m X'$ is also a borel set as it is a relative complement in $r_m X$. Since the Hausdorff Measure is a measure on borel sets, $\mathscr{H}^k(r_m X) = \mathscr{H}^k(r_m X \setminus r_m X') + \mathscr{H}^k(r_m X')$, which means that $\mathscr{H}^k(r_m X) = \mathscr{H}^k(r_m X \setminus r_m X')$. Since all $r_i X \setminus r_i X'$ are disjoint,

$$\mathscr{H}^k(X) = \sum \mathscr{H}^k(r_i X) = \mathscr{H}^k(X) \sum r_i^k,$$

which means that $\sum r_i^k = 1$, so k is equal to the similarity dimension. [Hut81] ■

6. Minkowski Dimension

6.1. **Definition.** Like the Hausdorff Dimension, the Minkowski Dimension, created by Hermann Minkowski and Georges Bouligand, is used to find the dimension of a set in metric space. However, it is much easier to compute. Unlike the Hausdorff Dimension, which uses balls with differing diameters, the Minkowski Dimension uses congruent shapes with the same diameter. The Minkowski Dimension is essentially the way the amount of shapes needed to cover the set scales with the size of the shapes. The mass of the fractal is the amount of shapes that cover it, and scaling up the fractal is done by scaling down these shapes. The n-dimensional Minkowski Dimension is defined in terms of a lower and upper limit, which when equivalent indicate the Minkowski Dimension, $\dim_{\mathscr{M}}(X)$.

The Lower Minkowski Dimension is:

$$\underline{\dim_{\mathscr{M}}}(X) = \liminf_{e \to 0} \frac{\log N(e, A)}{\log \frac{1}{e}}.$$

The Upper Minkowski Dimension is:

$$\overline{\dim_{\mathscr{M}}}(X) = \limsup_{e \to 0} \frac{\log N(e, A)}{\log \frac{1}{e}}.$$

Here, e is the size of the shape, and $N(e, A)$ is the amount of shapes of size e needed to cover A, the set being measured.

The Minkowski Dimension is often referred to as the Box-Counting Dimension, as $N(e, X)$ is defined as the number of grid n-cubes in Euclidean Space with sidelength e X intersects. However, that is often very hard to define in a metric space, so instead, open balls are used. $N(e, X)$ is defined as the minimal number of balls of diameter e needed to cover the set X.

6.2. **Examples.**

Example. If X is a finite collection of k elements, $\dim_{\mathscr{M}}(X) = 0$, as when e is less than the minimum distance between two of the elements, $N(e, X) = k$, as one ball is needed for every element since no ball can contain two. That means that the Minkowski Dimension is

$$\underline{\dim_{\mathscr{M}}}(X) = \overline{\dim_{\mathscr{M}}}(X) = \lim_{e \to 0} \frac{\log N(e, X)}{\log \frac{1}{e}} = \frac{k}{\infty} = 0.$$

Example. If X is the set $[0, 1]$,

$$\frac{1}{e} \leq N(e, X) \leq \frac{1}{e} + 1.$$

From this, we have

$$\underline{\dim_{\mathscr{M}}}(X) = \lim_{e \to 0} \frac{\log \frac{1}{e}}{\log \frac{1}{e}} = 1,$$

$$\overline{\dim}_{\mathscr{M}}(X) = \lim_{e \to 0} \frac{\log \frac{1}{e} + 1}{\log \frac{1}{e}} = 1.$$

Therefore, $\dim_{\mathscr{H}}(X) = 1$.

Proposition 6.1. *The Cantor Set C has a Minkowski Dimension of $\frac{\log 2}{\log 3}$. The Cantor Set is formed by starting with the set $[0,1]$, and at each iteration, removing the middle third of each line segment, making the first iteration $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.*

Figure 12. The formation of the Cantor Set [Source]

Proof. We can see that each iteration has 2^n segments of length 3^{-n}. Therefore, if our balls have diameter $3^{-n} < e \leq 3^{-(n-1)}$, at most 2^n balls will be needed, as each ball can cover one segment of length 3^{-n}. This gives us $N(e, C) \leq 2^n$ for this interval. This means that

$$\overline{\dim}_{\mathscr{M}}(C) = \lim_{e \to 0} \sup \frac{\log N(e, C)}{\log \frac{1}{e}} \leq \lim_{n \to \infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}.$$

Since every ball of diameter e can intersect at most two segments in an iteration, $N(e, C) \geq 2^{n-1}$, so

$$\underline{\dim}_{\mathscr{M}}(C) = \lim_{e \to 0} \inf \frac{\log N(e, C)}{\log \frac{1}{e}} \geq \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 3^n} = \lim_{n \to \infty} \frac{n-1}{n} \times \frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.$$

Since $\overline{\dim}_{\mathscr{M}}(C) \leq \frac{\log 2}{\log 3}$, $\underline{\dim}_{\mathscr{M}}(C) \geq \frac{\log 2}{\log 3}$, and $\overline{\dim}_{\mathscr{M}}(C) \geq \underline{\dim}_{\mathscr{M}}(C)$,

$$\underline{\dim}_{\mathscr{M}}(C) = \overline{\dim}_{\mathscr{M}}(C) = \dim_{\mathscr{M}}(C) = \frac{\log 2}{\log 3}.$$

■

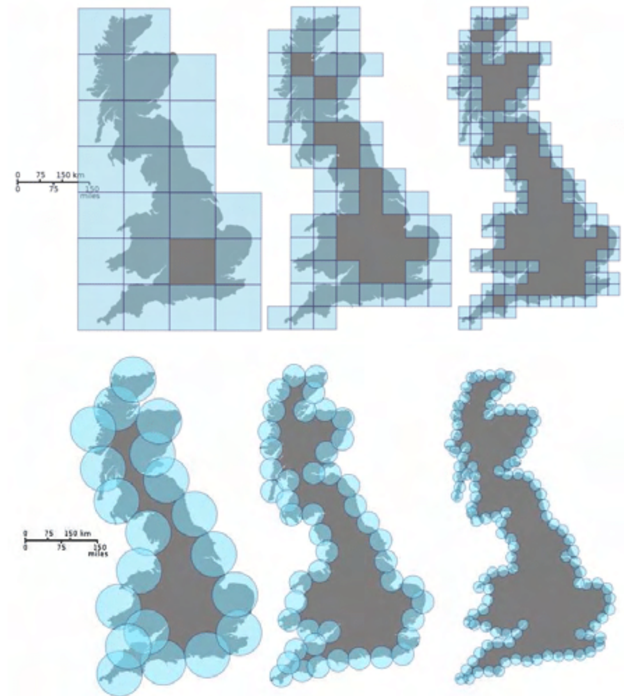
6.3. Relation to Hausdorff Dimension.

The Minkowski and Hausdorff Dimensions satisfy the inequality $\dim_{\mathscr{H}} \leq \underline{\dim}_{\mathscr{M}} \leq \overline{\dim}_{\mathscr{M}}$. Though they are usually equivalent, there are rare cases where the Minkowski Dimension can give a differring result – notably, with sets with countably infinite elements.

Proposition 6.2. *The set $K = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\}$ has a Minkowski Dimension of $\frac{1}{2}$.*

Proof. We know that the distance between $\frac{1}{m}$ and $\frac{1}{m-1}$ in K to be $\frac{1}{(m-1)(m)} > \frac{1}{m^2}$. With this, for any $1 > e > 0$, we can define an n such that $\frac{1}{(n+1)^2} < e \leq \frac{1}{n^2}$, which gives that $\frac{1}{\sqrt{e}} \geq n$.

Figure 13. Finding the Minkowski Dimension of the British Coastline using Box-Covering (Above) and Ball-Covering (Below). The coastline has a dimension of approximately 1.26. [Source 1] [Source 2]



We can see that the elements in $[1, \frac{1}{n}]$ need to be covered by n segments of length e, since for any element $1 \leq m \leq n$,

$$d(\frac{1}{m-1}, \frac{1}{m}) > \frac{1}{m^2} \geq \frac{1}{n^2} \geq e,$$

meaning that no two elements in this subset can be covered by one open interval of length e. For the elements in $[\frac{1}{n+1}, 0]$, they can be covered by at most $n+1$ open intervals, since because $e > \frac{1}{(n+1)^2}$, the number of intervals of length e to completely cover the interval of length $\frac{1}{n+1}$ is at most $\frac{\frac{1}{n+1}}{\frac{1}{(n+1)^2}} = n+1$. Therefore, we have

$$N(e, K) \leq n + n + 1 \leq 2e^{-\frac{1}{2}} + 1.$$

We also have that

$$N(e, K) \geq e^{-\frac{1}{2}},$$

since $e > \frac{1}{(n+1)^2}$, $e^{-\frac{1}{2}} < n+1$ and $N(e, K) \geq n+1$, as an additional e is needed to cover $[\frac{1}{n+1}, 0]$. Therefore, we have

$$e^{-\frac{1}{2}} \leq N(e, K) \leq 2e^{-\frac{1}{2}} + 1.$$

We can now calculate the Minkowski Dimension.

$$\underline{\dim}_{\mathscr{M}}(K) = \lim_{e \to 0} \frac{\log e^{-\frac{1}{2}}}{\log \frac{1}{e}} = \frac{1}{2},$$

and

$$\overline{\dim}_{\mathscr{M}}(K) = \lim_{e \to 0} \frac{\log 2e^{-\frac{1}{2}} + 1}{\log \frac{1}{e}} = \frac{1}{2}.$$

Therefore, the Minkowski Dimension of K is $\frac{1}{2}$. [BP17] ■

However, this doesn't agree with the Hausdorff Dimension. For any dimension greater than 0, the infimal covering will be made by only covering the points, such that each set has a diameter 0 and thus a volume of 0. Thus the Hausdorff Dimension is 0. Indeed, the Minkowski Dimension does not have the property mentioned in Theorem 4.18; The dimension of the union of a countable number of subsets has a higher dimension than their supremum.

7. Additional Definitions

Definition 7.1. Lebesgue Covering Dimension

An open cover of a set X in metric space is a set of open sets U_i such that $X \subseteq \bigcup U_i$. The order of such an open cover is the minimal integer m such that each point belongs to at most m sets in the cover. A refinement of an open cover is another open cover $\bigcup V_i$ such that every V_i is contained within some U_i. The Lebesgue Covering Dimension of X is the minimum n such that for every finite open cover there is an open refinement with order $n+1$.

8. Conclusion

Though the field of fractal theory is a relatively recent development in mathematics, there is still much more to explore, and this paper is only scratching the surface. There are many more ways to define fractal dimension other than the ones found here, such as the packing dimension and information dimension. In addition, many fractal sets are not trivial to find the dimension of, especially non-self-similar ones, and proving the fractal dimension of these often involves tools from other areas of math, leaving quite a few open problems. Finally, as mentioned at the beginning, fractal theory has plenty of use cases in the real world, leaving many opportunities for the application of this concept.

9. Acknowledgements

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