GALTON WATSON PROCESS

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ABSTRACT. The Galton Watson Process is **enter definition here**. In this paper, we go over the general definition and discuss three applications **insert applications here**

1. INTRODUCTION

Galton-Watson (GW) branching processes, including the invariant Galton-Watson (IGW) branching processes, have found applications in various fields, including biology, epidemiology, physics, and social sciences. Here are some notable applications of Galton-Watson (IGW) branching processes:

(1) Epidemic Modeling: GW branching processes have been used to model the spread of infectious diseases in populations. Each infected individual generates offspring (new infections) based on certain transmission rules. This modeling approach helps understand the dynamics of disease propagation and assess the effectiveness of interventions.
(2) Species Extinction: In ecology and evolutionary biology, GW branching processes have been employed to study the extinction of species. By considering population dynamics and reproductive rates, these processes provide insights into the probability of a species becoming extinct over time.
(3) Social Network Analysis: GW branching processes, especially the IGW variant, have been used to model information diffusion and cascades in social networks. By simulating the spread of information through branching processes, researchers can analyze the patterns and dynamics of information propagation in online social platforms.
(4) Genealogy and Family Trees: Galton-Watson processes have been used in genealogy research to model the growth and evolution of family trees. By considering reproductive patterns, these processes help understand the distribution and structure of family lineages.
(5) Particle Physics: In the study of particle physics, GW branching processes have been applied to model the decay and fragmentation of particles. By considering the branching behavior of particles, researchers can study the evolution of particle showers and cascades in high-energy physics experiments.
(6) Financial Mathematics: Galton-Watson processes have been used in mathematical finance to model the growth and fluctuation of asset prices. By incorporating branching mechanisms, these processes help analyze the risk and volatility of financial markets.

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(7) Queuing Theory: GW branching processes have been employed in queuing theory to model the arrival and service of customers in a system. By considering the branching nature of customer arrivals, researchers can analyze queue lengths, waiting times, and system performance.

1. 2. 3. 4. 5. 6.

2. Background

The origin of this model can be traced back to F. Galton's statistical investigation into the disappearance of family names. During the nineteenth century, there was a concern among the Victorians that aristocratic surnames were vanishing. In 1873, Galton initially raised this question about the probability of such an occurrence in an edition of "The Educational Times." The Reverend H. W. Watson later provided a solution in response. Subsequently, they collaboratively authored a paper titled "On the probability of the extinction of families" for the Journal of the Royal Anthropological Institute in 1874. Over time, this model has found diverse applications in biology (gene fixation, early evolution of bacterial colonies), chemistry (chemical chain reactions), and physics (cosmic rays). However, it is important to note that this model has limited usefulness in comprehending actual distributions of family names, as such names can change for various reasons beyond the extinction of a male family line.

3. Galton-Watson Process

A Galton-Watson (GW) Process is a more complex version of a Markov Chain. In simple words, it begins with a population and then evolves in discrete time following these rules:

(1) Each nth generation individual produces a random number of offspring for the next $(n+1)$st generation
(2) The offspring counts $\xi_\alpha, \xi_\beta, \xi_\gamma$ for individuals α, β, γ are mutually independent and are also independent from offspring counts or previous generations. These are also identically distributed with the distribution $p_{k k\geq 0}$
(3) the state Z_n is the number of individuals in the nth generation at time n

Definition 3.1. The transition probabilities are $PZ_{n+1} = k | Z_n = m = p_k^{*m}$ where the

4. Social Networks

Branching processes serve as discrete-time stochastic processes employed to model the evolution of populations over time. Recently, these have been used for studying how information spreads in large online user generated websites like Twitter and Reddit.

Understanding the factors that trigger information cascades holds significant implications both in scientific and commercial contexts. From a scientific perspective, the viral spread of content on platforms such as Twitter manifests as a visible outcome of complex collective behavior among uncoordinated and extensive masses.

Analyzing the propagation of information in-depth enables us to discern between content that captures user attention and content that elicits indifference. From a practical standpoint, users of web platforms often find themselves inundated with mostly unwanted content, potentially causing relevant and timely information to get lost in the shuffle. Therefore, accurately estimating which content will gain popularity becomes crucial in providing improved ranking and personalization of content.

The computation of the SPG (Stochastic Process of Growth) associated with a given node follows these steps:

(1) We begin with node i and assign its SPG a value of 1.
(2) For each neighboring node, denoted as j, a coin is tossed with a success probability of k. If the coin toss results in success, we select (retain) node j; otherwise, we discard it.

This process is repeated for each selected neighbor of i and is equivalent to drawing a sample of neighbors from i, where the sample size follows a binomial distribution. The number of trials equals the degree of i, and the success probability is k. The process restarts from the selected nodes: if node j has been chosen, then with a decreased probability of p, we continue the exploration into j's neighborhood. We will demonstrate that with an appropriate selection of nodes, the described process converges.

The SPG of node i is defined as the sum of all coefficients associated with the nodes in graph G that have been selected during the network traversal process outlined above.

We refer to the stochastic processes described as a graph-driven branching process. The primary contributions of this research are summarized as follows.

The graph-driven branching process represents a novel stochastic process. In standard branching processes, the probability of extinction, which refers to the probability of no individuals existing after a certain generation, solely depends on the expected number of children an individual can have. In our case, the extinction of the exploration process is jointly influenced by the function and the graph topology.

At generation k, a node j has an expected number of children equal to . Therefore, nodes with high degrees can compensate for lower fertility levels, while high fertility can overcome limitations stemming from nodes with low degrees, thereby averting early extinction of the process.

Additionally, it is worth noting that classical branching processes assume a constant probability for an individual to have a particular number of children across generations. In contrast, our model incorporates a fertility ratio that decreases as the generation counter k increases.

5. Earthquake Occurrence

Stochastic branching processes have been extensively employed for modeling earthquake occurrence since the 1970s, with influential contributions by Kagan (1973), Kagan Knopoff (1976), and Vere-Jones (1976). One widely used approach builds upon the self-exciting Hawkes point process (Hawkes 1971; Adamopoulos 1976; Daley Vere-Jones 2003), which is equivalent to a branching process with immigration (Hawkes Oakes 1974; Saichev et al. 2005; Baró 2020).

The earthquake population can be divided into clusters, each originating from an immigrant event and including all subsequent offspring and their offspring.

These clusters can be represented as tree graphs, where the root corresponds to the initial immigrant event, the other vertices represent triggered earthquakes, and the edges represent triggering relationships dictated by the model (which may or may not align with actual physical triggering if fitted to data).

The inclusion of immigrants (background events) ensures the population does not diminish over time. From 1970 to 1990, powerful probabilistic tools were developed to handle space-time-magnitude generalizations of branching processes, establishing a tradition of seismological applications (Vere-Jones, Ogata, et al.).

In the context of earthquake modeling, each earthquake generates offspring according to a modified Omori law (Omori 1894; Utsu 1970; Utsu et al. 1995). Specifically, an earthquake with magnitude M_i occurring at time t_i produces offspring, termed first-generation aftershocks, according to a Poisson process with an intensity given by the following equations.

$$\nu(t|t_i, M_i) = \frac{K_0 10^{\alpha(M_i - M_0)}}{(t - t_o + c)^p} t > t_i \tag{5.1}$$

This equation incorporates positive constants κ_0, α, c, and $p > 1$. Every new event, in this case representing earthquakes, is assigned an independent magnitude M_i, independent from any parents. These are typically calculated according to the Gutenberg-Richter law (Gutenberg Richter 1944) expressed by equation (2).

Definition 5.1 (Gutenberg-Richter Law)**.** The number of earthquakes with magnitude M is proportional to 10^{-bM} where b is a positive number that can vary but is overall equal to $b = 1$

The overall earthquake flow consists of background events, their first-generation aftershocks, second-generation aftershocks (offspring of the first generation), third-generation aftershocks (offspring of the second generation), and so on. This combined flow can be represented as a point process characterized by its conditional intensity $(t|H_t)$ defined as.

$$(t|H_t) = \mu(t) + \sum_{i:t_i<t} \nu(t|t_i, M_i) \tag{5.2}$$

Here, H_t denotes the process history, which comprises the events (t_i, M_i) occurring before time t. It is important to note that this modeling framework can include spatial components by incorporating a point field with a background intensity (t, x) and a conditional space-time distribution of offspring given by a density $\nu(t, x|t_i, x_i, M_i)$ (Ogata 1998, 1999).

A Galton-Watson (GW) stochastic branching process describes a population that starts with a single progenitor at step $s = 0$ and evolves in discrete steps. At each time step, each existing member independently gives birth to $k = 0, 1, ...$ offspring according to a distribution p_k, without considering other members, and then terminates.

By focusing solely on parent-offspring earthquake relations in the ETAS model, without considering time and space attributes, a single cluster that originates from an earthquake of random magnitude and includes all generations of aftershocks can be described by a GW process.

Definition 5.2 (Critical Process)**.** If the average offspring number is unity ($\sum_k p_k = 1$)

Definition 5.3 (Subcritical Process)**.** If the average offspring number is less than unity ($\sum_k p_k < 1$)

A critical GW process exhibits a population size of unity on average at each step, while a subcritical GW process experiences exponential decline in the average progeny.

Critical and subcritical GW processes generate finite populations with probability 1, although the average size of a critical population is infinite.

Within the Epidemic Type Aftershock Sequence (ETAS) model framework, the offspring distribution p_k is obtained by considering the conditional Poisson distribution of offspring numbers for a parent earthquake with a given magnitude and integrating it with respect to the magnitude distribution (2).

The article proposes a theoretical modeling framework for earthquake clustering based on a family of invariant Galton-Watson (IGW) branching processes.

IGW processes serve as rigorous approximations to imprecisely observed or inaccurately estimated earthquake clusters modeled by GW branching processes, ETAS model. The theory of IGW processes provides explicit distributions for various cluster attributes, including magnitude-dependent and magnitude-independent offspring numbers, cluster size, and cluster combinatorial depth. The study demonstrates the close fit between the IGW model and observed earthquake clusters by analyzing seismicity in southern California. The estimated IGW parameters and derived statistics prove robust, even when considering different lower cut-off magnitudes in the earthquake catalog. The proposed model enables analysis of multiple seismicity quantities based on self-similar tree attributes and facilitates the assessment of seismicity proximity to criticality.

Definition 5.4 (Invariant Galton-Watson Process (IGW))**.** A critical GW process with an offspring distribution given by $q_1 = r$ and recursively defined

$$q_k = (1-r)\frac{(1-q)(k-\frac{1}{q})}{q(2-\frac{1}{q})k!}$$

The power index $\frac{1+q}{q}$ decreases from 3 to 2 with the model parameter q increases from $\frac{1}{2}$ to 1.

In other words, the offspring distribution q_k has generating function

$$Q(z) = \sum_{k=0}^{\infty} q_k z^k = z + (1-r)q(1-z)^{\frac{1}{q}} \tag{5.3}$$

where $q \in [\frac{1}{2}, 1)$ and $r \in [0, 1)$

6. Conclusion

In conclusion, Galton-Watson (GW) branching processes, specifically the invariant Galton-Watson (IGW) branching processes, offer a theoretical framework for understanding the dynamics of information spread in social networks. These processes have gained popularity as a tool for studying how information propagates in large online social networks such as Twitter and Reddit. The IGW branching processes provide a mathematical description of how information cascades occur in social networks. Each member of the population generates offspring based on predefined rules, leading to the formation of clusters or cascades of information propagation. These cascades can be represented as tree graphs, where the root corresponds to the initial information source, and subsequent vertices represent individuals who receive and propagate the information. By employing GW branching processes, researchers can analyze various aspects of information spread, such as the factors triggering cascades, the patterns of attention and indifference among users, and the challenges of content ranking and personalization. The model allows for a better understanding of the complex collective behavior of uncoordinated masses in social networks and provides insights into discriminating between attention-catching information and indifferent information. The application of GW branching processes to social networks offers both scientific and practical implications. At the scientific level, it helps uncover the mechanisms underlying information propagation and the interplay between network topology and cascade dynamics. From a practical standpoint, it aids in addressing the challenge of content overload and improving the ranking and personalization of content for users. While the GW branching processes provide valuable insights into information spread in social networks, it is important to note that they have certain limitations. The applicability of these models in understanding actual information distribution patterns is limited because factors other than the extinction of information cascades can lead to changes in the distribution of content in social networks. In summary, Galton-Watson (IGW) branching processes serve as a useful tool for studying information propagation in social networks. They offer insights into the factors triggering cascades, user attention, and the challenges of content ranking. However, it is crucial to consider the limitations of these models in capturing the complete dynamics of information spread in real-world social networks.

7. References

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