The Stokes Theorem

Multivariable Calculus

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Double Integrals

Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

- $$\int_a^b \int_c^d f(x,y)\,dy\,dx$$
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- We can use the same method with polar co-ordinates to calculate the double integrals of some functions, as it is easier in situations with circular symmetry.

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Vector Fields

A field which assigns a vector $\mathbb{F}(x, y)$ to a point (x,y), in $\mathbb{R}^2$ or $\mathbb{R}^3$

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- Vector Fields in $\mathbb{R}^3$

$$F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$$

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- A line integral allows us to calculate quantities along this curve by considering the contributions from the vector field that we encounter as we move along the curve
- $$\int_C f(\mathrm{r})\, ds$$
- We can use line integrals to calculate the work done by a force along a specific path, among other things.

Surface Integrals

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$$\mathrm{div}(F) = \nabla \cdot F$$

- Curl measures the rotational behavior of a vector field, quantifying how much a vector field curls or circulates around a given point.

$$\mathrm{curl}(F) = \nabla \times F$$

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$$\iint_R (curl \times \mathrm{F}) \cdot d\mathrm{A} = \oint_C \mathrm{F} \cdot dr$$

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- It allows us to calculate line integrals discussed earlier, by converting them into double integrals.

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- It states that for a vector field $F = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$ in $\mathbb{R}^3$, where P, Q, and R have continuous first-order partial derivatives, the line integral of F around a closed curve C is equal to the surface integral of the curl of F across any surface S bounded by C

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Stokes Theorem

$$\int_C \mathrm{F} \cdot d\mathrm{r} = \iint_S \mathrm{curl}\,\mathrm{F} \cdot d\mathrm{S}$$

- Through this theorem, line integrals can be evaluated using the simplest surface with boundary C.

Stokes Theorem continued.

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- Let S be a surface, and D be a segment small enough that it can be broken into square A (four sides listed below) . Using the green's theorem, if we compute the flux of this square:

$$\int_A (\mathrm{A}_l + \mathrm{A}_d + \mathrm{A}_r + \mathrm{A}_u) \cdot d\mathrm{r} = \iint_A \operatorname{curl} \mathrm{F} \cdot \mathrm{N}\, dS = \iint_A \operatorname{curl} \mathrm{F} \cdot d\mathrm{S}$$

Stokes Theorem continued.

- We now approximate the flux over the entire surface (purpose of stokes theorem) and this can be done using the green's theorem (albeit in a long calculation!) After simplification:

Remember

$$\iint_R (curl \times \mathrm{F}) \cdot d\mathrm{A}$$

so,

$$\iint_D [-(Ry - Qz)zx - (Pz - Rx)zy + (Qx - Py)]dA$$

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Remember

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so,

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- This basically equals:

$$\iint_S \operatorname{curl} \mathrm{F} \cdot d\mathrm{S}$$

Proving the Faraday's Theorem

Theorem: *The voltage change around a loop is proportional to the negative of the time rate of change of the magnetic flux through the loop.* Then, voltage V through a curve segment x is:

$$\Delta V \approx E \cdot \Delta x$$

Hence the change in voltage around the loop is:

$$\int_C E \cdot dx$$

Thank You!