THE GENERALIZED STOKES THEOREM

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Abstract. This paper will focus on the Stokes theorem, envisioned in 1850 by William Thomson. The aim is to progress from calculations involving double integrals to line and surface integrals. The paper will also highlight the fundamental theorem of calculus and the existence of the Stokes theorem and Green's theorem. The discussion will also involve a proof of Faraday's law through the stokes theorem, establishing one of its application in physics.

Contents

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1. Fundamental theorem of Calculus

The fundamental theorem of calculus, establishes the basis of integration and anti-derivatives such that the theorem comes into use when integrating in different forms, and using various methods, much like those discussed in this paper.

Theorem 1.1. Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x)$ be an antiderivative of $f(x)$, i.e., $F'(x) = f(x)$. Then, we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

2. Double Integrals

Double integrals are second-order integrals, which refer to integrals of any two variables, as opposed to one in a simple integral. They are used to calculate the total bounded area of the average value part of a function in a defined two-dimensional area (given there are two variables). In this paper, we consider the case for general regions.

Firstly, we must establish the types of regions one can consider.

Defination 2.1 A region P in the (x, y) plan is a **Type 1** region if it lies between the graph of two continuous functions and two verticl lines, with the following parametrization

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

A region P in the (x, y) plan is a **Type 1** region if it lies between the graph of two continuous functions and two horizontal lines, with the following parametrization:

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Theorem 2.1 To calculate a double integral over a general region R, we can use an iterated integral. Let $f(x, y)$ be a function defined on R. The double integral can be expressed as:

$$\iint_R f(p, q)\,dA = \int_a^b \int_c^d f(p, q)\,dp\,dq$$

a and b represent the limits of integration for the variable p, and c and d are used to present the limits of integration for q. The inner integral $\int_c^d f(p, q)\,dq$ is evaluated with p held constant, and the outer integral $\int_a^b \int_c^d f(p, q)\,dp\,dq$ integrates the subsequent result over the specified range.

Example 2.1

$$V = \int_0^1 \int_0^2 (8x + 6y)\,dx\,dy = \int_0^1 (4x^2 + 6xy\Big|_{x=0}^{x=2})\,dy = \int_0^1 (4 + 6y)\,dy = 4y + 3y^2\Big|_0^1 = 20$$

After establishing the basics of double integrals over a general region, we can use it to calculate an average over general functions, and similar applications including a density function with information about mass (m)

Let us now consider the case of double integrals over polar coordinates, a case where it helps in calculating the integrals due to the geometric defination of the curve/area

Theorem 2.2 The double integral of the function $f(r, \theta)$ over the same general region described in Theorem 2.1, D, in the $r - \theta$ plane is defined as,

$$\iint_D f(x,y)\, r\, dx\, dy = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Integration over polar co-ordinates has wide scale applications in helping calculate the volume and area of radial structures/surfaces, being much easier than doing it through the method we discussed in Theorem 1.11

Example 2.2 Following is an example of double integrals through polar co-ordinates over a trigonometric function.

$$\begin{aligned}
\iint_R (r^2 \sin(\theta))\, r\, dr\, d\theta &= \int_0^{\frac{\pi}{4}} \int_0^2 (r^3 \sin(\theta))\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\frac{r^4}{4} \sin(\theta) \Big|_0^2 \right) d\theta = \int_0^{\frac{\pi}{4}} \frac{16}{4} \sin(\theta)\, d\theta \\
&= 4 \int_0^{\frac{\pi}{4}} \sin(\theta)\, d\theta = 4 \left(-\cos(\theta) \Big|_0^{\frac{\pi}{4}} \right) \\
&= 4 \left(-\cos\left(\frac{\pi}{4}\right) + \cos(0) \right) = 4 \left(-\frac{1}{\sqrt{2}} + 1 \right) \\
&= 4 - 2\sqrt{2}
\end{aligned}$$

Double integrals over polar co-ordinates are also useful in calculating masses, weighted average and moments of inertia.

Example 2.2 Calculating moments of inertia
A moment of inertia is defined as the equivalent of mass for any rotational motion, with the rotation calculating inertia through polar co-ordinates becomes easier.

To find the moment of inertia about an axis for a solid with density δ, denoted as I_0, we can use the formula $I_0 = \iint_R r^2 \delta\, dA$, where R represents the region of integration.

Let's consider the case of a disk with radius a around its center ($\delta = 1$). To find the moment of inertia about the x-axis, we can set up the integral as follows:

$$I_x = \iint_R y^2 \delta\, dA$$

We then change variables to polar coordinates with the origin at a point. Assuming the diameter of the disk is along the x-axis, the polar equation of the circle is $r = 2a\cos(\theta)$. Therefore, the integral becomes:

$$I_0 = \iint_R r^2 \, r \, dr \, d\theta = \ldots = 2\pi a^4$$

With these double integrals we can understand that there are various applications for integrals as we increase the area we define the integrand on. Further we must understand the existence of vector fields to better understand the space these areas exist in.

3. Vector Fields

Moving further into the applications we can look at an extension of integral concepts, in the form of vector fields.

A vector field is the definition of a vector to each point in any given space space, most commonly an Euclidean space.

Definition 3.1 A vector field $\mathbf{F}$ in $\mathbb{R}^2$ is a two-dimensional vector with relation $\mathbf{F}(x, y)$ to each point (x, y) of a smaller subset D of $\mathbb{R}^2$. The subset D acts as the domain of the vector field.

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

In both cases, the vector field assigns a vector to each point in the given subset, representing the magnitude and direction of the field at that point. The domain D specifies the region in which the vector field is defined and can be any subset of the respective coordinate space.

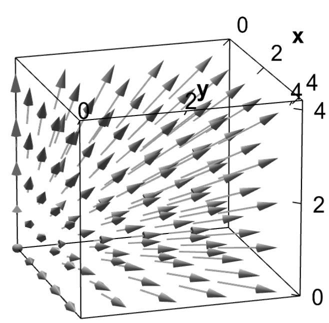


Figure 1. Sample Vector Field

Examples of common applications of vector fields, include calculating the air flow around an airfoil and ocean currents. To calculate the airflow we represent the velocity of the airflow at each point in space as a vector, and construct a vector field that describes the airflow pattern, simmilar to the example 3.1.2 3.1

3.1. **In $\mathbb{R}^2$.** A vector field in $\mathbb{R}^2$ is a function F that to each point (x, y) is a vector $F(x, y)$

$$\begin{aligned}\mathbf{F}(x,y) &= P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} \\ &= \langle P(x,y), Q(x,y)\rangle\end{aligned}$$

Example 3.1.1
The vector field $\mathbf{F}(x, y) = (3x^2 + 2y, \sin(y))$ is a continuous vector field in $\mathbb{R}^2$.

To find the vector associated with the point $(2, -1)$, we substitute the coordinates into the components of the vector field:

$$\mathbf{F}(2,-1) = (3(2)^2 + 2(-1), \sin(-1)) = (12, -\sin(1))$$

Example 3.1.2
Application of $\mathbb{R}^2$ vector fields in calculating the velocity of a fluid. We must consider the vector field that models its velocity and consider a specific point, to calculate its instantaneous velocity, in this case at point (2,-2)

Suppose that $\mathbf{v}(x, y) = (-2yx^2 + y^2)\mathbf{i} + (2x^3 + y^2)\mathbf{j}$ is the velocity field of a fluid.
To find the velocity at point $(2, -2)$, insert the point into $\mathbf{v}$:

$$\mathbf{v}(2,-2) = \left(-2(-2)(2)^2 + (-2)^2\right)\mathbf{i} + \left(2(2)^3 + (-2)^2\right)\mathbf{j} = \mathbf{i} + \mathbf{j}.$$

The velocity of the fluid at this point is the calculated value of this vector. Therefore, it is $\|\mathbf{i} + \mathbf{j}\| = 40\,\mathrm{m/s}$.

3.2. **In $\mathbb{R}^3$.** We can represent vector fields in $\mathbb{R}^3$ with component functions, simply needing an extra component function for the extra dimension, represented by:

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

They have similar applications to vector fields in $\mathbb{R}^2$, simply in a 3d space in this case.

3.3. **Gradient fields.** A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

$$\frac{\partial f}{\partial x} = P(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = Q(x,y)$$

$$\frac{\partial f}{\partial x} = P(x,y,z), \quad \frac{\partial f}{\partial y} = Q(x,y,z), \quad \text{and} \quad \frac{\partial f}{\partial z} = R(x,y,z)$$

These equations identify the requirements for a vector field to be a gradient field

4. Line Integrals

In the various integrals discussed above, we have been integrating over line segments. However, if we aim to integrate over any curve in the plane, and not just a line segment in one axis, we must use *line integrals.*

Line Integrals allow for integration over a vector field or curve in a plane or 3 dimensional space. Simmilarly, Surface Integrals 5 allow for integration over an entire surface rather than a path, which is the case with line integrals.

Definition 4.1 The line integral of a given function $f(x, y)$ along C is denoted by:

$$\int_C f(a,b)\,ds$$

ds refers to the movement along the curve rather than an axis, making it the line integral of the function f over the arc length of C:

$$\int_C f(a,b)\,ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{da}{dt}\right)^2 + \left(\frac{db}{dt}\right)^2}\,dt$$

Example 4.1 If we consider the line integral $\int_C xy^4 ds$ where C is the right part of the function $x^2 + y^2 = 4$, we must parameterize it through polar coordinates:

$$x = r\cos(\theta) = 4\cos(\theta)$$
$$y = r\sin(\theta) = 4\sin(\theta)$$

To solve ds,

$$dx = -4\sin(\theta)\,d\theta$$
$$dy = 4\cos(\theta)\,d\theta$$

Substituting these values into the line integral, we have:

$$\int_C xy^4\,ds = \int_0^\pi (2\cos(\theta))(2\sin(\theta))^4$$

$$= \sqrt{(-2\sin(\theta))^2 + (2\cos(\theta))^2}\,d\theta$$

Simplifying the integrand, we get:

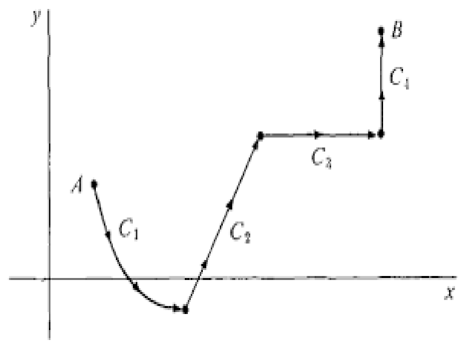
$$\int_C xy^4\,ds = \int_0^\pi (2\cos(\theta))(2\sin(\theta))^4$$

$$= \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)}\,d\theta$$

$$\int_0^{\pi} 16\cos(\theta)\sin^4(\theta)\,d\theta$$

$$= 8\pi^2 - 4\pi^4 + \frac{16}{6}\pi^6$$

Theorem 4.1 Line integrals are calculated over a parametrization, however, can be calculated individually and their sum is considered in the case of a piecewise curve, like the one below:



$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds + \int_{C_4} f(x,y)\,ds$$

This process makes it much easier to calculate line integrals, even those in a 3 dimensional space.

Line integrals have various applications including calculating the mass of a wire, or the flux in a vector field. They are very important in understanding the Stokes Theorem **??**, as these integrals establish the basis of vector calculus. An extended form of line integrals is surface integrals which will be discussed next.

5. Surface Integrals

Surface integrals are a generalization of multiple integrals to integrate over entire surfaces. Unlike, line integrals the integration in this case is done over the entire surface as opposed to a simple path, due to which we must parametrize the surface.

Definition 5.1 Given the parameterization of the surface $\mathbf{D}(a,b) = \langle x(a,b), y(a,b), z(a,b)\rangle$, the parameter domain is the set of points in the ab-plane that can be substituted into $\mathbf{D}$.

Theorem 5.1 For any given surface $z = g(x,y)$, following is the formula for the surface integral.

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dA \tag{5.1}$$

There is a clear difference between line integrals and surface integrals, and while the aspect of parametrization might not be easy to understand, a visualisation of both will certainly help. The following figure gives a clear understanding of both, with the region on the left representing a line integral, and the right representing a surface integral.

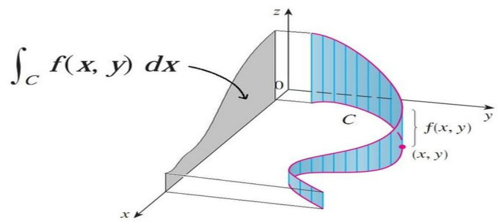


Figure 2. Line and Surface Integrals

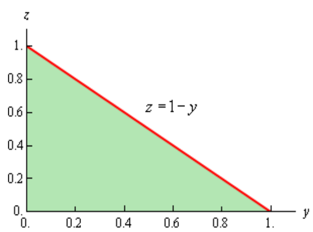
Example 5.1

To better understand the concept of surface integrals and their difference to line integrals, let us explore an example:

$$\iint_S 6xy\, dS$$

Let's calculate the given integral over an octant of the plane $x + y + z = 1$

This area/surface on the plane can be visualized as the following



This gives us the following limits in the y and z plane, we then input in the formula 5.1:

$0 \leq y \leq 1$

$0 \leq z \leq 1 - u$

$$\iint_S 6xy\, dS = \int_0^1 \int_0^{1-u} 6uv\sqrt{3}\, dv\, du.$$

Integrating with respect of v and inserting limits gives:

$$\sqrt{3} \cdot 6u \left[\frac{(1-u)^2}{2} - 0 \right] = \sqrt{3} \cdot 6u \cdot \frac{(1-u)^2}{2}.$$

Integrating with respect to u and simplifying:

$$\sqrt{3} \cdot 3 \int_0^1 u(1-u)^2\, du.$$

To solve this integral, we can use basic integration:

$$\sqrt{3} \cdot 3 \left[\int_0^1 u\, du - 2\int_0^1 u^2\, du + \int_0^1 u^3\, du \right] = \sqrt{3} \cdot 3 \left[\frac{1}{12} \right] = \frac{\sqrt{3}}{4}.$$

The example above clearly shows the implementation of theorem 5.1 and the difference between a surface integral and line integral. In theory, this difference is also the major difference/relation between The Green's Theorem 6 and The Stokes Theorem **??**

5.1. **Divergence and Curl.** To understand the theorem we will explore further in this paper, we must accustom ourselves to the concepts of divergence and curl, which are operators and are used to describe behaviour of integrals in vector fields.

Definition 5.1.1 *Divergence* is an operator which defines how a vector field behaves towards or away from a point. Eg: In electromagnetism, the divergence of the electric field vector represents the presence of electric charges, and their movement away/towards a certain point.

Theorem 5.1.1 The divergence of a vector field $F = (F_x, F_y, F_z)$ is:

$$\text{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{5.2}$$

or can be alternatively be written using a gradient field 3.3

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} \tag{5.3}$$

Definition 5.1.2 *Curl* is an operator which helps define the rotation of a vector field about a certain point. Eg: It helps describe the rotational behavior of the vector field and as such can be used to determine the circulation of any fluid or the presence of vortices in a flow.

Theorem 5.1.2 The curl of a vector field $F = (A, , B, C)$ in $\mathbb{R}^3$ where $A_x, A_y, A_z, B_x, B_y, B_z, C_x, C_y, C_z$ all exist, is:

$$\text{curl } \mathbf{F} = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \frac{\partial A}{\partial z} - \frac{\partial A}{\partial x}, \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y} \right) \tag{5.4}$$

Both these concepts will be extremely useful as we move forward and explore 6 and 7.

6. The Green's Theorem

As discussed earlier, green's theorem is an extension of the fundamental theorem of calculus in a higher dimension. Green's theorem basically relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C.

It also allows us to calculate line integrals discussed earlier, by converting them into double integrals, this makes calculations much simpler compared to the cumbersome method described in 4

Theorem 6.1 Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region containing D, then

$$\oint_C (P(x, y)\, dx + Q(x, y)\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

where $\oint_C$ represents the line integral around the curve C, and $\iint_D$ represents the double integral over the region D. Curve C:

Green's theorem can be used only for a two-dimensional vector field F.

$$\int_C P\,dx + Q\,dy = \int_C F \cdot T\,ds$$

where $\int_C$ represents the line integral around the curve C, P and Q are the components of the vector field F, dx and dy are the differentials of x and y respectively, T is the unit tangent vector to C, and ds is the differential arc length along C.

Example 6.1 Let us calculate the line integral with radius 2, using Green's theorem

$$\oint_C (3x^2 + 4xy)\,dx + (2y + x^2)\,dy$$

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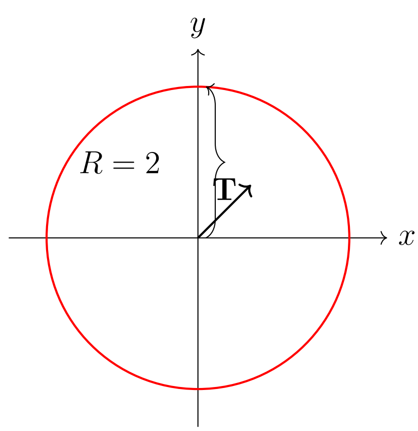

Using Green's theorem, we have:

$$\begin{aligned}\oint_C (3x^2 + 4xy)\,dx + (2y + x^2)\,dy &= \iint_D \left(\frac{\partial(2y + x^2)}{\partial x} - \frac{\partial(3x^2 + 4xy)}{\partial y} \right) dx\,dy \\ &= \iint_D (2 - 4x)\,dx\,dy,\end{aligned}$$

where D is the region enclosed by the circle C. Since the region D is symmetric, we can simplify to:

$$\begin{aligned}\iint_D (2-4x)\,dx\,dy &= 2\iint_D dx\,dy - 4\iint_D x\,dx\,dy \\ &= 2\times \text{Area}(D) - 4\times 0 \\ &= 2\times \pi R^2 \\ &= 2\pi(2^2) \\ &= 8\pi.\end{aligned}$$

To understand the concept better, on can try the following **Example 6.2** :

$$\int_C 2y\,dx + 3x\,dy$$

As shown in the first example 6this calculation becomes much simpler because we use the green's theorem if we were to solve it directly, the calculation would be much longer. A similar case for surface integrals is seen when we move onto the stokes theorem.

7. The Stokes Theorem

The theorem was initially developed by Lord Kelvin, who communicated the result to George Stokes in a letter (1850), however, is named after the latter to honour his work in the field of fluid dynamics and mathematical analysis.

Theorem 7.1 If there is a surface S with n number of pieces with a closed boundary of the curve C and if $\mathbf{F}$ is a vector field with components that have continuous partial derivatives on an open region containing S, then:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.$$

where $\int_C$ represents the line integral along the curve C, $\int_S$ represents the surface integral over the surface S, $\mathbf{F}$ is the vector field, $d\mathbf{r}$ is the differential displacement vector along the curve, $\operatorname{curl}\mathbf{F}$ is the curl of the vector field $\mathbf{F}$, and $d\mathbf{S}$ is the differential surface area vector on the surface.

Note: As mentioned above, the stokes theorem relates the curl operator and the vector field F.

Let us look at a proof for the stokes theorem, we do this by examining the concept of line and surface integrals 4,5

First, let us look at the LHS of the stokes theorem 7

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Parametrizing the curve C:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}, \mathbf{F}\cdot\mathbf{n}, dS,$$

where $\mathbf{n}$ is the outward unit normal vector to the surface S and dS represents the differential element of the equation Now, recall that the curl of a vector field is defined as 5.1. Using the this, we can rewrite the surface integral as a volume integral:

$$\iint_S \text{curl}, \mathbf{F} \cdot \mathbf{n}, dS = \iiint_V (\nabla \times \mathbf{F}) \cdot d\mathbf{V},$$

where V is the region enclosed by the surface S. Now, let's consider the RHS:

$$\iint_S \text{curl}, \mathbf{F} \cdot d\mathbf{S}.$$

We can now rewrite the surface integral as a line integral:

$$\iint_S \text{curl}, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

We end up with the same result on both sides, giving us a logical proof for the Stokes theorem. While there is a longer and more comprehensive proof for the Stokes theorem, it is beyond the scope of this paper, and can be further researched upon. Stokes theorem has various applications which make it extremely useful in both math and physics, one such application is of that in proving the faraday's law.

7.1. **Proof for Faraday's Law.** *Faraday's law* relates the curl of an electric field to the rate of change of the corresponding magnetic field (the negative rate of change). If we are to use the stokes theorem to prove Faraday's law, we must first recall its mathematical relation.

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S},$$

This relates the line integral of electric field E along closed loop C to negative time derivative of surface integral of magnetic field B over surface S.

By applying Stokes' theorem, we equate the line integral to the surface integral:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S},$$

where $\nabla \times \mathbf{E} = \text{curl}$.

Now, we equate this to magnetic flux (negative in the case of faraday's law):

$$-\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

Since this equation holds for any surface S bounded by the closed loop C, we can equate the integrands:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This equation relates the curl of the electric field to the rate of change of the magnetic field with respect to time, which is the purpose of Faraday's Law. With a simple set of calculations we were able to derive the relation using the stokes theorem, highlighting the strength of its applications.

Conclusion

This paper focused on building the foundational concepts leading up to the Stokes' theorem. There are several additional topics regarding Stokes' theorem and its applications which should be interesting to explore for the reader, such as using the theorem to relate integration and mappings. The Stokes' theorem has many applications in physics, other than the discussion of faraday's theorem in this paper, and this paper provided a basis to understand these powerful applications.

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