Quaternions and Algebras

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What is a quaternion?

A quaternion is a number of the form $q = t + xi + yj + zk$ for real numbers t, x, y, z.

An octonion is a number of the form $x_\infty + \sum_{n=0}^{6} x_n i_n$ where the i_n satisfy $i_n^2 = -1$ and

$$i_{n+1} i_{n+2} = i_{n+4} = -i_{n+2} i_{n+1}$$

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with indices taken modulo 7.

Historical background

They were invented by William Rowan Hamilton in 1843. A plaque on the Broom Bridge in Dublin, Ireland commemorates his discovery of the equation $i^2 = j^2 = k^2 = ijk = -1$.

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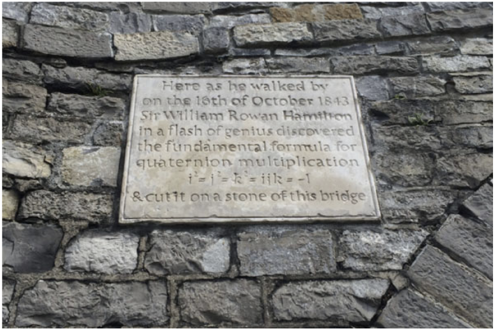

Quaternion algebras

A field is a set combined with addition and multiplication operations with additive and multiplicative identities 0 and 1. Quaternion algebras need to have a basis $1, i, j, ij$ satisfying $i^2 = a$, $j^2 = b$, and $ij = -ji$.

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2. $\overline{\overline{\alpha}} = \alpha$ for all $\alpha \in B$; and
3. $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$ for all $\alpha, \beta \in B$.

Quadratic Forms

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Definition 1

A similarity from Q to another quadratic form $Q' : V' \to F$ is a pair (f, u) where $f : V \tilde{\to} V'$ is an F-linear isomorphism and $u \in F^\times$ satisfy $Q'(f(x)) = uQ(x)$ for all $x \in V$. An isometry is a similarity with $u = 1$.

Quadratic Forms

Definition 2

A quadratic form is a map $Q : V \to F$ on an F-vector space V satisfying:

1. $Q(ax) = a^2 Q(x)$ *for all* $a \in F$ *and* $x \in V$; *and*
2. *The map* $T : V \times V \to F$ *defined by*

$$T(x, y) = Q(x + y) - Q(x) - Q(y)$$

   is F-bilinear.

Characteristic 2

Well, then our algebra B would be defined as follows:

- $i^2 + i = a$
- $j^2 = b$
- $ij = j(i + 1)$

Simplest algebras?

A simple algebra is one that only has $\{0\}$ and itself as two-sided ideals.
A representation of B is a vector space V together with a homomorphism from B to the set of endomorphisms from V to F.
A representation is given by a left or right B-module.

Hurwitz integral quaternions

A Hurwitz quaternion is a quaternion with t, x, y, z integers or half-integers. The Hurwitz units include the basis units, their negatives, and the quaternions with each component $1/2$ or $-1/2$.

Factoring Hurwitz integers

A primitive Hurwitz integer can be factored into Hurwitz primes with factorization unique to unit-migration.

What are the only composition algebras?

Theorem 3

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$ *are the only composition algebras.*

This theorem also proves the nonexistence of an algebra of higher dimension, a proposed algebra with 16 dimensions as a result of extending the octonions.

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- $[a + ib, c + id] = [a, c] + [b, d]$
- $\overline{a + ib} = \bar{a} - ib$
- $(a + ib)(c + id) = (ac - d\bar{b}) + i(cb + \bar{a}d)$

Applications

Quaternions were invented by Hamilton to model 3D rotations. They are practical because they can be stored with less memory than a rotation matrix. The use of quaternions also avoids gimbal lock, which is when two of the three rotation axes line up.