The Hadwiger-Nelson Problem: A Colorful Journey Through The Plane and Beyond

Samarth Das
samarthdas870@gmail.com

Euler Circle

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Definition (Filter)

A filter on a set X is a collection $\mathcal{F}$ of subsets of X(i.e. $\mathcal{F} \subseteq P(X)$) satisfying the following criteria:

1. $X \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$.
2. If $Z \subseteq Y \subseteq X and Z \in \mathcal{F}$, then $Y \in \mathcal{F}$.
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The idea is to take f in this case is the chromatic coloring function, thus we can show that since a finite graph is a subset of the infinite plane their chromatic numbers must be equivalent by the ultrafilters.

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Now we can prove the bounds.

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$$\chi(\mathbb{E}^2) \geq 4.$$

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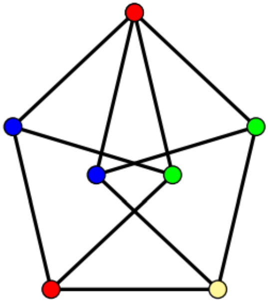

Figure: The Moser Spindle.

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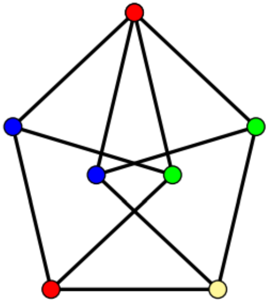

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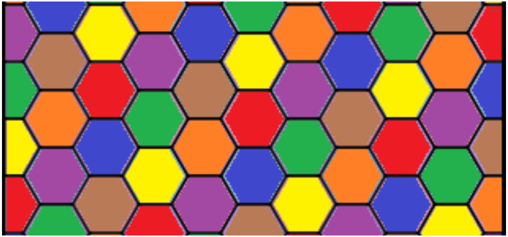

Figure: The 7-chromatic hexagonal tiling.

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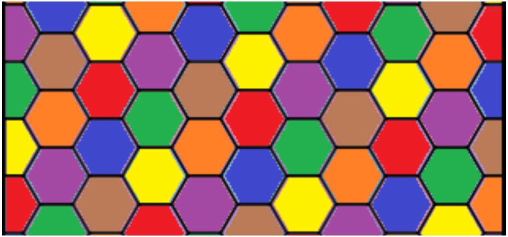

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Is This Really The Lower Bound?

I may have lied. I apologize, but the lower bound isn't actually 4. We have Aubrey De Grey to blame for that. In fact, the lower bound is actually 5, as we will see in his novel construction.

De Grey's Construction

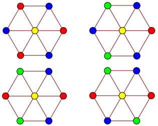


Figure: A cell and its various colorings.

Definition (Monochromatic Triple)

A monochromatic triple is a set of 3 vertices that are colored the same color.

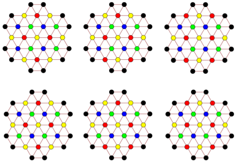


Figure: A honeycomb and its various colorings.

Definition (Linking Vertices)

Linking vertices are the vertices at distance 2 from the center.

De Grey's Construction

Definition (Linking Diagonal)

A linking diagonal is a pair of linking vertices located in opposite directions from the center.

Definition (Specific Coloring)

A specific coloring is a coloring such that two opposite linking vertices are the same color as the center and all the other four are a second color.

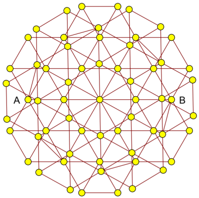
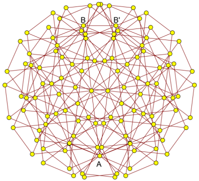


Figure: A hive.



De Grey's Construction

The pictures of the large graphs are not shown due to visual impenetration. The idea is to start from a hexagonal cell and construct a swarm graph, which contains 52 copies of the cell that cannot have a monochromatic linking diagonal. This then allows us to perform the same operation on a moser wheel, which produces the solution.

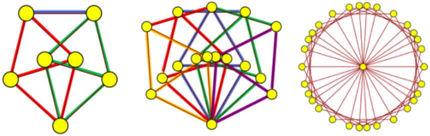


Figure: A Moser Wheel.

I'm tired of this problem. Can I have another one?

The Hadwiger-Nelson problem has a rich history, which led to many further explorations. We will discuss two of them:

- Polychromatic Number of The Plane
- Chromatic Number of The n-dimensional Space

Polychromatic Number

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We can in fact bound χ_p as we did for χ.

Theorem (Raiskii's Theorem)

$$\chi_p \geq 4$$

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This bound is not as simple as the Moser Spindle construction, although the proof is quite nice.

Lower Bound

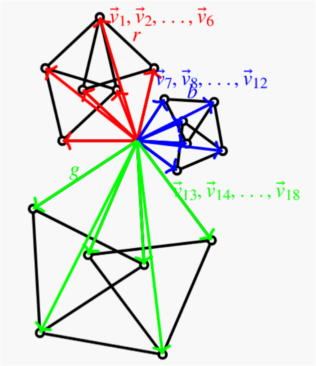

Figure: Merkov's Construction

The idea is to use a proof by contradiction on this specific construction. We count the number of arrangements of red, blue, and green vertices and show that it exceeds the total number of arrangements of vertices.

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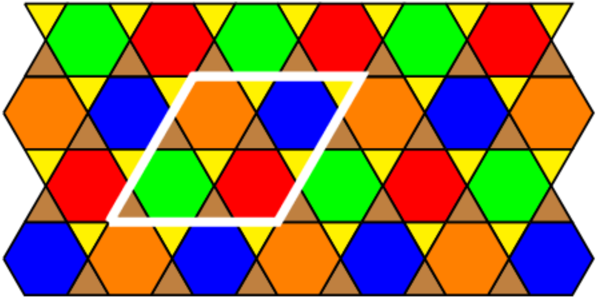

Figure: Steichkin's Pattern.

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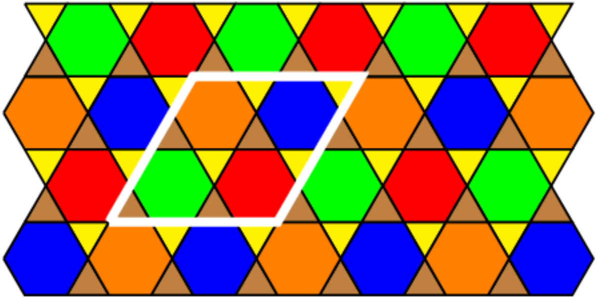

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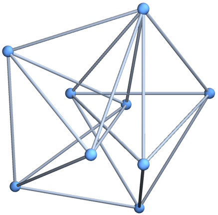

Figure: The 3-diamond construction.

Method

Replace the double triangles in Moser Spindles with double n-simplices.

More Than The Plane

Theorem (The Upper Bound)

$$\chi(\mathbb{E}^n) \leq \lfloor 2 + \sqrt{n} \rfloor^n.$$

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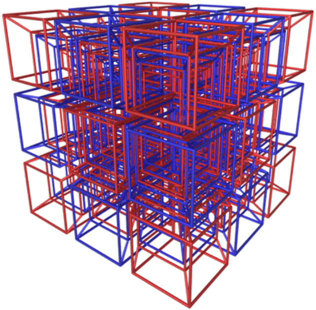

Figure: A Tesseractic tiling.

Method

Size the tesseracts such that their diagonals have lengths less than 1 and two neighboring tesseracts have their monochromatic points at least 1 apart.

Conclusion

The Hadwiger-Nelson Problem is a quite interesting and mind-boggling exploration. It has captivated the minds of mathematicians throughout the twenieth century and beyond. Perhaps you can make a difference; read my paper for more information.

Here are some questions to stimulate your thinking:

Extension (Three Dimensions)

Can you find better bounds for the $n = 3$ case of the n-dimensional chromatic number of the space?

Extension (Rational Coloring)

If we only consider rational points on the plane, do the bounds still remain inequalities or can we find explicit values? (Hint: $\mathbb{R} > \mathbb{Q}$.)

Credits

Thank you! Any questions?

(Bonus) The Masterpiece

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