THE HADWIGER-NELSON PROBLEM: A COLORFUL JOURNEY THROUGH THE PLANE AND BEYOND

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ABSTRACT. The Hadwiger-Nelson problem is a famous unsolved problem in mathematics that concerns the field of graph theory, in particular coloring the plane. The question asks the following:

Open Question 0.1 (The Hadwiger-Nelson Problem)**.** *How many colors are needed to color the points of the plane such that no two points exactly* 1 *unit distance apart are assigned the same color?*

In other words, it asks for the value of $\chi(\mathbb{E}^2)$, where χ is the chromatic number and $\mathbb{E}^2$ denotes the Euclidean plane. This problem has given rise to many further explorations and discoveries, some of which we will discuss in this paper.

1. Introduction

The Hadwiger-Nelson Problem has a rich history behind it. It was originally attributed to Nelson, who first recorded asking the question in 1950, yet over the years it had caught the names of many other prominent mathematicians such as Erdos. While it is ambigious who was the first to establish the problem, the work behind it can be explained in a much more structured manner, which we will explore. [11]

2. Infinite Plane? Oh No

It is natural to ask how this question would be approached, since it considers $\chi(\mathbb{E}^2)$, which by definition is an infinite set of points. Fortunately with the use of the graph theoretical De Bruijn–Erdős Theorem, the plane can be reduced to a finite graph. (Note that there exists a incidence geometrical De Brujin-Erdos Theorem , but unfortunately for us it is irrelevant in this context.) Its statement is as follows:

Theorem 2.1 (De Bruijn–Erdos's Compactness Theorem)**.** *An infinite graph G is k-colorable if and only if every finite subgraph of G is k-colorable.*

To prove this theorem, we will first need a few definitions:

Definition 2.2 (Filter)**.** A filter on a set X is a collection $\mathcal{F}$ of subsets of X(i.e. $\mathcal{F} \subseteq P(X)$) satisfying the following criteria:

(1) $X \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$.
(2) If $Z \subseteq Y \subseteq X and Z \in \mathcal{F}$, then $Y \in \mathcal{F}$.
(3) If $Y, Z \in \mathcal{F}$, then $Y \cap Z \in \mathcal{F}$.

Definition 2.3 (Ultrafilter)**.** An ultrafilter on X is a filter on X that contains as many sets as possible. (a maximization of $\mathcal{F}$.)

Date: July 17, 2023.

Definition 2.4 (Axiom of Choice)**.** Every family ϕ of nonempty sets has a choice function, i.e., there exists a function f such that $f(S) \in S$ for every S from ϕ.

Proof. Let X be the collection of all finite subsets of V. In other words,

$$X = W \subseteq V \mid W \text{ is finite}$$

. For each $W \in X$, let $X_W = \{U \in X \mid W \subseteq U\}$, and let $\mathcal{F}$ be the set of all $Y \subseteq X$ such that, for some $W \in X, X_W \subseteq Y$.

Note that it is trivial that $\mathcal{F}$ is a filter on X by simply checking the definition conditions listed in **Definition 2.2**. Thus we can immediately apply the Axiom of Choice to find an ultrafilter $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}$.

For every $W \in X$, we can find a function $f_W : W \to d$ such that, for all $v, w \in W$, if $\{v, w\} \in E$, then $f_W(v) \neq f_W(w)$. In other words, we can find a chromatic coloring $f_W : W \to \{0, 1, \ldots, d-1\}$.

For $w \in V$ and $i < d$, let

$$X_{w,i} = W \in X \mid w \in W \text{ and } f_W(w) = i$$

. The characteristics of $\mathcal{G}$ as an ultrafilter guarantee the existence of a unique $i_w < d$ such that $X_{w,i_w} \in \mathcal{G}$. Now all we have to do is to show that this is unique to all the other i_w. Let us formalize this argument by defining a function $f : V \to d$ by letting $f(w) = i_w$ for all $w \in V$.

We claim that f is a chromatic coloring, i.e., for all $\{u, v\} \in E$, we have $f(u) \neq f(v)$.

Proof. Fix $\{u, v\} \in E$. Since $\mathcal{G}$ is an ultrafilter we can find a set W in $X_{u,i_u} \cap X_{v,i_v}$. Now we must have $u, v \in W, f_W(u) = i_u$, and $f_W(v) = i_v$. Since f_W is a chromatic coloring and $\{u, v\} \in E$, it follows that $i_u \neq i_v$ and hence $f(u) \neq f(v)$. ■

■

This theorem essentially boils down the problem to finite graphs, which is much easier to tackle as we see. As this result had been proven before the Hadwiger-Nelson Problem became mainstream, it was immediately known that the bounds on $\chi(\mathbb{E}^2)$ were $4 \leq \chi(\mathbb{E}^2) \leq 7$. We will exhibit constructions of each bound in the section below.

3. Proving The Bounds

In this section we consider the proofs of the lower and upper bounds. These follow from explicit constructions shown here rather than a more rigorous proof.

Theorem 3.1 (Classic Lower Bound)**.**

$$\chi(\mathbb{E}^2) = 4.$$

Proof. Assume for the sake of contradiction that $\chi(\mathbb{E}^2) = 3$. Then all we need to do is work our way downwards, from which we find that at least one edge is monochromatic, contradiction. This specific graph is called the Moser Spindle, a nice construction in its unit-distance form which we will generalize later. ■

Theorem 3.2 (Classic Upper Bound)**.**

$$\chi(\mathbb{E}^2) = 7.$$

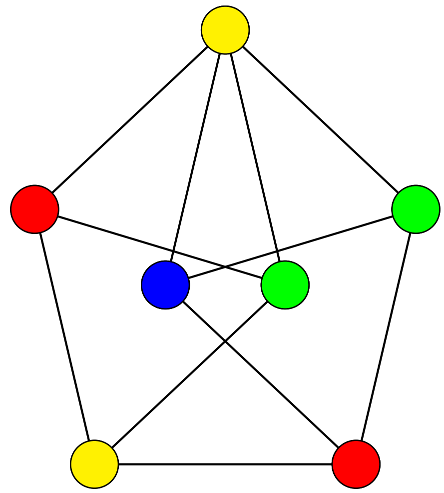

Figure 1. The Moser Spindle.

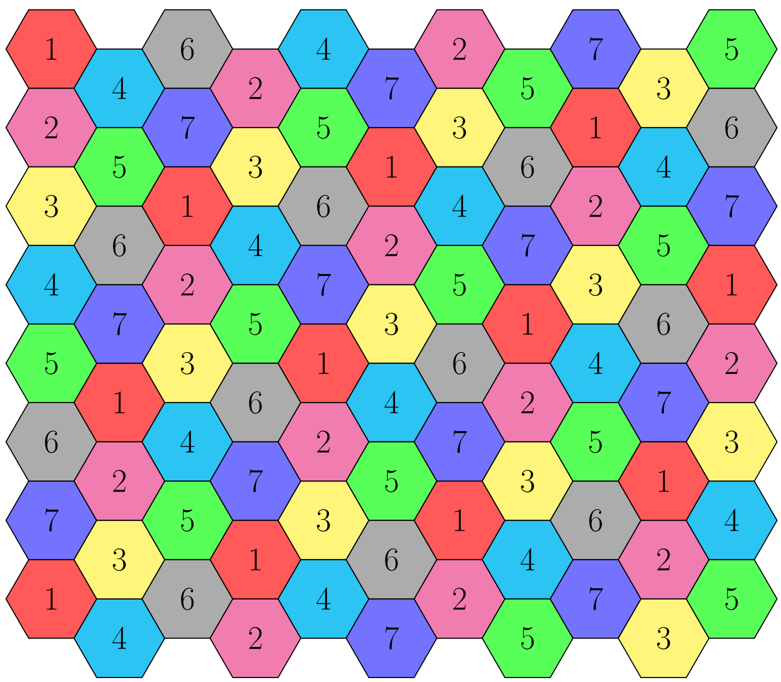

Figure 2. A 7-coloring of the plane.

Proof. It is well known that a plane can be tiled by regular hexagons of side length 1. Consider any hexagon. It is surrounded by 6 other hexagons, thus these 7 regions must all be different colors. It forms a flower-like arrangement of 7 hexagons. This argument works on every hexagon on the plane, thus concluding our proof. However, we must remember that monochromatic unit distances are prohibited. Fortunately, we can simply scale up the tiling by a small factor k, such as 1.1, which will increase distances and thus satisfy the problem conditions.

$\blacksquare$

4. Is This Really The Lower Bound?

For many years after the Moser Spindle and Golomb graphs were discovered, the problem was laid to rest. However, in 2018, de Grey discovered a 1581 vertex graph that required at least 5 colors. We discuss more details as follows.

5. Constructing a 5-chromatic unit-distance graph

We begin with some definitions and then dive straight into the construction.

Definition 5.1 (Monochromatic Triple)**.** A monochromatic triple is a set of 3 vertices that are colored the same color.

Definition 5.2 (Linking Vertices)**.** Linking vertices are the vertices at distance 2 from the center.

Definition 5.3 (Linking Diagonal)**.** A linking diagonal is a pair of linking vertices located in opposite directions from the center.

Definition 5.4 (Specific Coloring)**.** A specific coloring is a coloring such that two opposite linking vertices are the same color as the center and all the other four are a second color.

Construction 5.5 (Cell)**.** To construct this graph, the main idea is to start small; from the basic hexagonal 7-vertex, 12-edge graph and look at the different ways to color it. We see that the bottom two do not have a monochromatic triple, which is what we will use in the following constructions as the base graph. Note that we disregard rotations and reflections and color transpositions in all constructions.

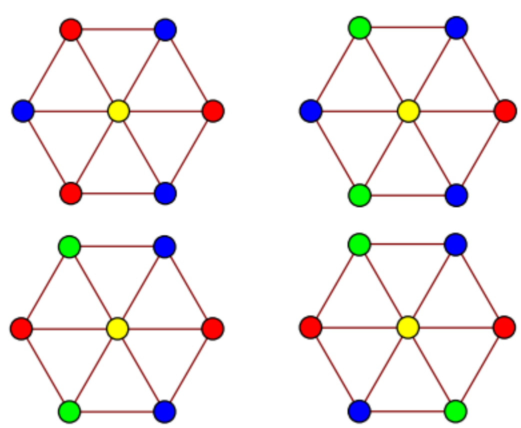

Figure 3. A cell and its various colorings.

Construction 5.6 (Honeycomb)**.** We construct a honeycomb graph that contains 13 copies of the honey graph. The monochromatic property reduces all the possibilities to essentially only 6 distinct cases, which are shown below. Note that the black vertices can represent any color as long as they are distinct from their neighboring vertices. We can now see that the two graphs on the right side have a specific coloring. We will focus on these graphs in the next construction.

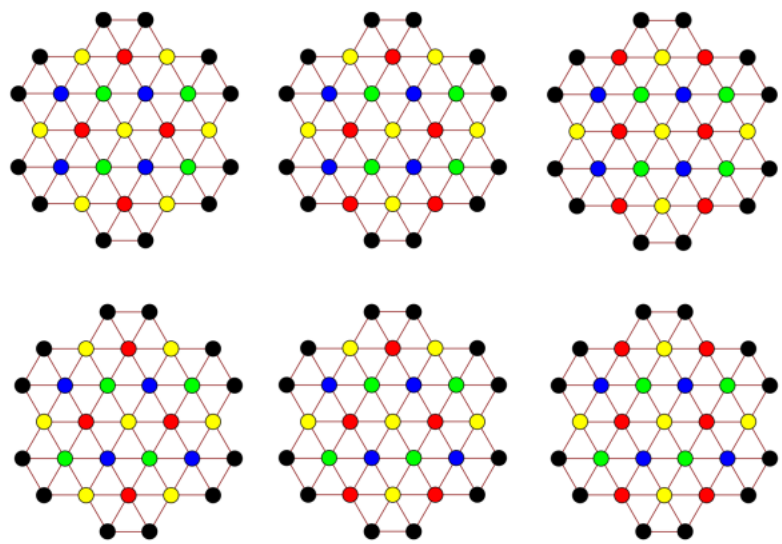

Figure 4. A honeycomb and its various colorings.

Construction 5.7 (Hive)**.** We construct a graph made of two copies of the honeycomb but one copy is rotated exactly $2\arcsin\frac{1}{4}$ about the center. The objective is to preserve the corresponding unit distances between points, allowing us to color both copies in a similar monochromatic fashion to the regular honeycomb graph. In fact, this means that since the specific coloring regular honeycomb graph has 3 monochromatic linking diagonals, the double honeycomb graph must have $3 \cdot 2 = 6$ monochromatic linking diagonals.

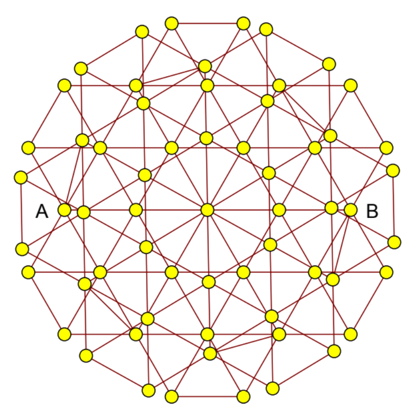

Figure 5. A hive.

Construction 5.8 (Swarm)**.** We now construct a graph made of two copies of the hive by $2\arcsin\frac{1}{8}$, but rotated about a specific point A denoted in Figure 5. Once again this is done to preserve unit distance. However, this rotation actually translates the previously linking diagonal to a unit distance, so thus it presents a contradiction from the hive graphs. In fact, it means that the points must be different colors, so we no longer have a monochromatic diagonal in the hive and thus at least one cell must be non monochromatic.

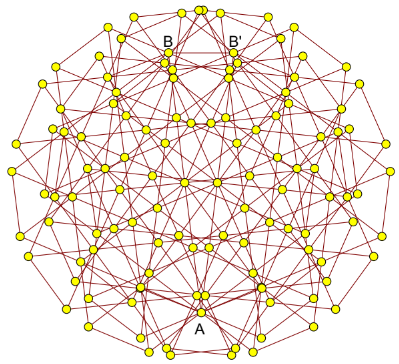

Figure 6. A swarm.

We have shown that there exists a graph that forces at least one monochromatic equilateral triangle - now it remains to show that we can block this property and force a fifth color via another construction.

The main idea is to return to Moser spindles - since they have the property that the two diagonals $\sqrt{3}$ apart are not both monochromatic. This property can be used in conjunction with our swarm construction in order to force 5 colors.

Construction 5.9 (Moser Wheel)**.** We can tightly link Moser Spindles to form the Moser Wheel as shown below. The idea of this construction is to preserve a central cell, which then can be proved non-monochromatic. Note that all edges are now $\sqrt{3}$ in length since these distances are preserved under transformations.

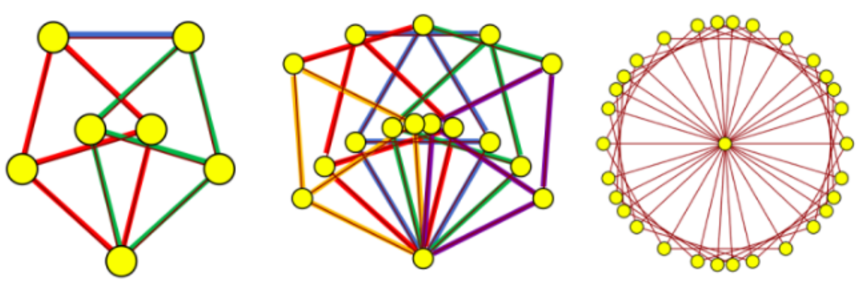

Figure 7. A Moser Wheel.

The following two graphs are shown in the Appendix due to visual impenetrability, yet somewhat nice-looking specimens.

Construction 5.10 (Mo(n)s(t)er)**.** Finally, we construct this graph which consists of 6 copies of the sum of two edges of the Moser Wheel. Essentially, we construct another graph such that its edges are the vector sum of the edges of the Moser Wheel, and whose magnitude is $\leq \sqrt{3}$. We then copy and translate it to form the monster, which clearly contains a large central cell as shown.

Construction 5.11 (Swarmonster)**.** Now that we have a graph that satisfies the conditions, it can be used in place of our cell in the first part of the group. Essentially, we proved that the construction worked for a cell as the base starting point. Now that we created a graph made entirely of Moser Spindles that must have opposite diagonals non-monochromatic, while in the cell they are allowed to be monochromatic, this will create a contradiction when we apply the transformations. Thus when we apply the swarm configuration to the monster we get a solution that satisfies the conditions.

6. Extensions

In the following sections, we will present several extensions of the Hadwiger-Nelson problem. These pose more interesting questions, some of which have a bounty (!).

7. Polychromatic Number of The Plane

After learning about the Hadwiger-Nelson Problem that only forbids unit distances, it is natural to extend this to other distances. In other words, for any color i, it is assigned a distance d_i which is forbidden; no two vertices may be exactly d_i apart. The problem becomes forbidding all these distances. Formalizing the question, we have:

Extension 7.1 (Polychromatic Number of The Plane)**.** *What is the smallest number of colors needed for coloring the plane in such a way that no color realizes all distances?*

We can denote this number as $\chi_p(\mathbb{E}^2)$, so that the original problem is the case $p = 1$. It is in fact possible to bound χ_p as well, in a similar fashion to χ. It is inherent that $\chi_p \leq \chi \leq 7$, as forbidding multiple distances would cause a lower number than forbidding only one distance. Unfortunately, this argument does not suffice for the lower bound. However, we can still provide a proof as follows.

Theorem 7.2 (Raiskii's Theorem)**.** *We have that*

$$\chi_p \geq 4$$

.

Proof. Assume FTSOC that there exists a 3-coloring of the plane $c : \mathbb{E}^2 \to \{\cdot, \cdot, \cdot\}$ such that the following holds:

- There are no two points colored red at the distance r from each other.
- There are no two points colored blue at the distance b from each other.
- There are no two points colored green at the distance g from each other.

We then produce the following construction:

The reason for doing this is to work in in the 18-dimensional vector space $\mathbb{E}^{18} = \{(a_1, a_2, \ldots, a_{18}) : a_1, a_2, \ldots, a_{18} \in \mathbb{R}\}$ and define a coloring $c' = (a_1, a_2, \ldots, a_{18}) = c(P)$ where P is the terminal point of the vector $a_1 \cdot v_1 + \ldots + a_6 \cdot v_6 + a_7 \cdot v_7 + a_12 \cdot v_12 + a_13 \cdot v_13 + a_18 \cdot v_18$.

Then we define a set $M \in \mathbb{E}^{18}$ such that the following conditions hold:

- $a_i \in \{0, 1\} \forall i \in \{1, 2, \ldots, 18\}$
- $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \in \{0, 1\}$
- $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} \in \{0, 1\}$

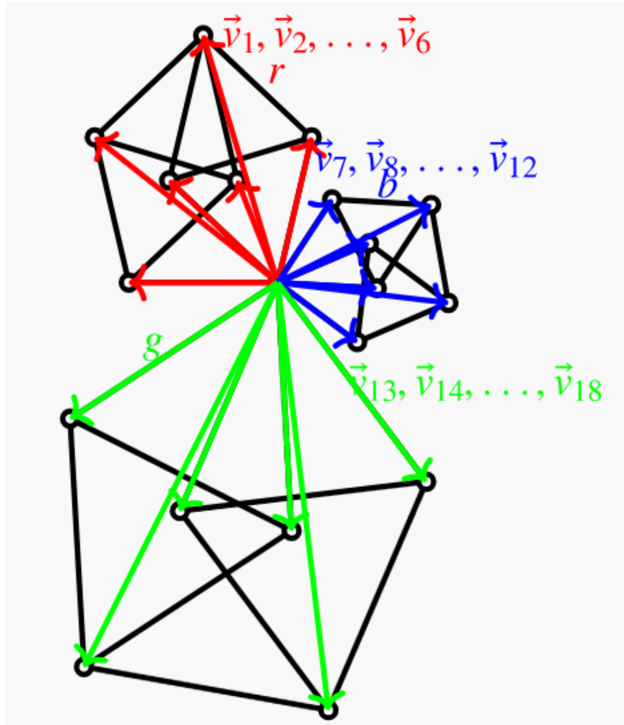

Figure 8. Merkov's Construction

- $a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} \in \{0, 1\}$

The cardinality of set M is $7^3 = 343$ which can be seen. The reason for dividing it into such a set is so that we can consider each Moser Spindle at a time. In particular, taking the red points, denoted $M_r = (a_1, a_2, \ldots, a_6, 0, 0, \ldots, 0)$, we can identify the following properties:

- The Moser Spindle with all edges of length r cannot have 3 red vertices
- The set M_r can only have two elements colored by the vector coloring .

Then we consider the complement of set M_r, denoted $M_{bg} = (0, 0, \ldots, 0, a_7, a_8, \ldots, a_{18}$. We can in fact translate M_r on top of this graph M_{bg} as chromatic number is preserved under translation. The key idea of this is that it allows us to count the number of M_r, in particular it is $7^2 = 49$ because each of the Moser Spindles has 7 colorings. Since no other elements can be colored red, we simply have that the number of these is the same as the number of red elements in the whole graph, an upper bound of $49 \cdot 2 = 98$. This construction also suffices for the blue and green, so we have that the total upper bound is $98 \cdot 3 = 294$. However, this is less than 343, the total number of colorings, so 294 cannot be an upper bound! Thus we are done, and there must be at least 4 colors used.

∎

For the upper bound, Stechkin found an explicit example:

Theorem 7.3 (Stechkin's Theorem). *We have that*

$$\chi_p \leq 6$$

.

Proof. We can tesselate the plane with a paralellogram pattern made up of equilateral triangles and regular hexagons. The hexagons can be colored with four colors, similar to the method in which we colored a square tiling, while the triangles are colored based on their orientation (up facing triangles are one color and down facing triangles are another color.) This uses a total of only 6 colors, and no color realizes all distances due to the fact that each vertex colored the same color is more than 1 or $\frac{1}{2}$ apart. Hence why it is called a $(1, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$ coloring. People have found other examples as well, which are shown in the appendix. ■

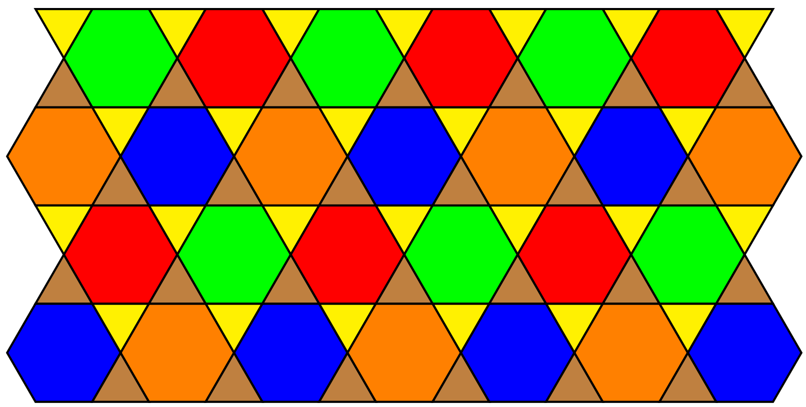

Figure 9. Steichkin's Pattern.

8. More Than The Plane

Another natural question to ask is what happens if we increase the dimensions, considering $\chi(\mathbb{E}^d)$ for an arbitrary dimension d. This turns out to have nice similarities to the $d = 2$ case, most notably the $d = 3$ case. We begin with a few definitions:

Definition 8.1 (Simplex)**.** A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

Definition 8.2 (Diamond)**.** We define a diamond as two copies of a n-simplex attached at their facets in opposite orientations.

We can immediately present a lower bound by relating it to the Moser Spindle.

Theorem 8.3 (Raiskii's Lower Bound)**.** *We have*

$$n + 2 \leq \chi(\mathbb{E}^n)$$

.

Proof. This follows from the "n-diamond construction." It is essentially a generalization of the Moser Spindle to n dimensions. It works by taking two "diamonds" and attaching them at one apex to the same vertex, and the other pair of apex by an edge of length 1. An example for the case $n = 3$ is shown below. ■

In a similar way, we can obtain an upper bound by relating it to the 2-dimensional square tiling of the plane.

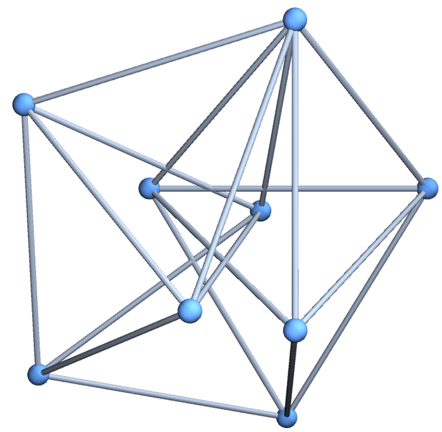

Figure 10. The 3-diamond construction.

Theorem 8.4 (Wikipedia's Upper Bound)**.** *We have*

$$\chi(\mathbb{E}^2) \leq \lfloor 2 + \sqrt(n) \rfloor^n.$$

Proof. This follows from the tiling of space by n-dimensional hypercubes and coloring it with k^n colors, where the pattern iterates after k colors in any direction. Note that we must account for the same problem we accounted for in the $n = 2$ case: the hypercubes must have edge length $\frac{1}{\sqrt{n}}$ to prevent diagonals from having unit distance, but more than $\frac{1}{k-1}$ so that two hypercubes of the same color are not unit distance apart. This gives us the following:

$$\frac{1}{k-1} < \frac{1}{\sqrt{n}} \Rightarrow k > 1 + \sqrt{n}.$$

Setting $k = \lfloor 2 + \sqrt{n} \rfloor$ is the smallest number for which this is satisfied. An example of a 4-dimensional hypercube tiling is shown below. ∎

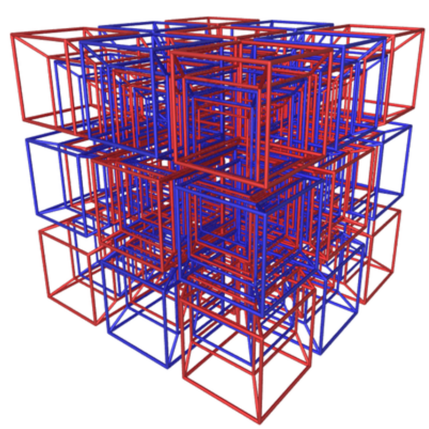

Figure 11. A Tesseractic tiling.

9. Three Dimensions

We begin with some definitions.

Definition 9.1 (Permutahedron)**.** A permutahedron of order n is an $(n-1)$-dimensional polytope embedded in an n-dimensional space. Its vertex coordinates are the permutations of the first n natural numbers. The edges identify the shortest possible paths that connect two vertices. Two permutations connected by an edge differ in only two places and the numbers on these places are neighbors. The permutahedron of order n has $n!$ vertices, each of which is adjacent to $n-1$ others. The number of edges is $\frac{(n-1)n!}{2}$, each with length $\sqrt{2}$.

As noted above, this is a specific case of the n-dimensional case, but produces very nice results. In fact, it allows us to greatly improve bounds:

Theorem 9.2 (Radoichic's Upper Bound)**.** *We have that*

$$\chi(\mathbb{E}^3) \leq 15.$$

The upper bound is once again proved by a construction as follows:

Proof. We can tile space with the truncated octahedron. The reason for using this specific shape is that it is a permutahedron of order 4, which allows us to separate unit distances. Note that the truncated octahedron has 24 vertices and 14 faces, 8 of which are regular hexagons and 6 of which are squares. This specific shape works because when we take the truncated octahedron with side length $\frac{1}{\sqrt{10}}$, opposite vertices are at distance 1. Then we simply just have to show that it can be colored with 15 colors in this fashion without unit distances. This is easy to do by simply checking all cases, since the solutions are of the form $5i + 3j + k \equiv 0 \pmod{15}$, where i, j, k are the unit vectors. A picture of the truncated octahedron and its tesselation are shown below. ■

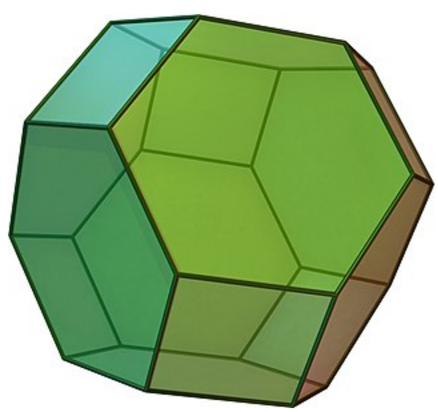

Figure 12. A truncated octahedron.

Now we prove the lower bound, which as in the 2-dimensional case is somewhat more rigorous, although by far not as lengthy.

Theorem 9.3 (Nechustan's Lower Bound)**.** *We have that*

$$6 \leq \chi(\mathbb{E}^3).$$

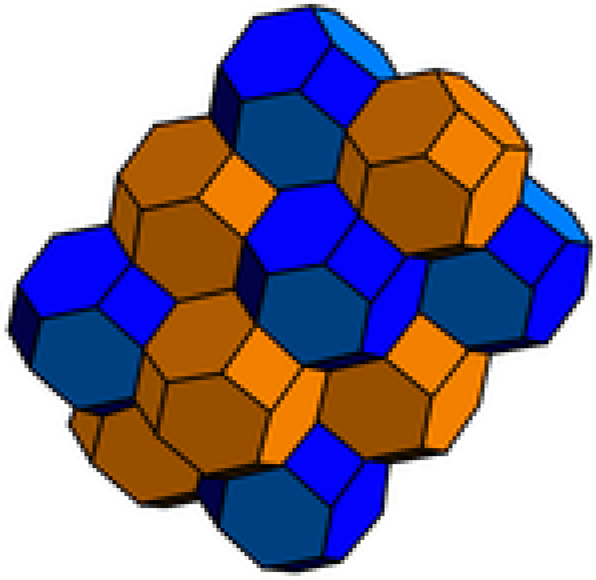

Figure 13. A truncated octahedron tesselation.

Proof. We make a nice construction as follows. Let s, t be two arbitrary points unit distanced apart in $\mathbb{E}^3$ and let $C = C_{s,t}$ be the circle of points that are unit distance from both s and t. Fix a sequence of distinct points $(p, p1, p2, q)$ on C that satisfy $|p - p1| = |p1 - p2| = |p2 - q| = 1$. Now let τ be a rotation of the space around the line $l = l(p, q)$ defined by the condition $|\tau(p_i) - p_i| = 1$ for $i = 1, 2$. Let G be the unit distance graph over $s, t, p, p1, p2, q, \tau(s), \tau(t), \tau(p_1), \tau(p_2)$.

Then we identify the following properties for a proper 5-coloring of G:

- Whenever (p, q) is monochromatic then neither $(p, p2)$ nor $(p1, q)$ as well as neither $(\tau(p), \tau(p_2))$ nor $(\tau(p_1), \tau(q))$ are monochromatic.
- Exactly one among $(p, q), (p, p2), (p1, q)$ as well as one among $(\tau(p), \tau(q), (\tau(p), \tau(p_2)), (\tau(p_1), \tau(q))$ is monochromatic.

This allows us to force a sixth color; i.e. we can show that if $\mathbb{E}^3$ is properly 5-colored, then rotating the circle $C = C_{p_1,\tau(p_1)}$ around the line $l(p, q)$ avoids the color $c(q)$, which is different from any point $c(p)$. This is easily shown by the properties we noted above. Now by simply taking $|p-q| = \frac{5}{3}$ and $|p-p_2| = |p_1-q| = \sqrt{8}3$, we achieve the desired result. The reason for these numbers is that the rotation formed entirely encloses the Raiskii Spindle as we showed above, which by definition has 5-colorings, so we are done.

■

10. Two Forbidden Distances

This is also a specific case of the polychromatic number noted above, and also produces nice results.

We define an additional parameter of the graph as follows:

Definition 10.1 (Two Forbidden Distances)**.** For a given positive real number $d \neq 1$, a $1, d$-graph is a finite graph whose vertices are points in the Euclidean plane E^2, and whose edges are obtained by connecting two points whenever the distance between them is either 1 or d.

Hereafter we exhibit certain examples of graphs that can be obtained using various values of d. Note that we will state the theorems here, but the diagrams will be included in the Appendix to maximize the viewing experience.

The following two theorems provide the cases of $d = 2$ and $d = \frac{2}{\sqrt{3}}$.

Theorem 10.2. $\chi(\mathbb{E}^2, \{1, 2\}) \geq 5$.

11. Rational Colorings

Another natural question to ask is what happens when we restrict the domain; instead of taking all $\mathbb{R}^2$ in $\mathbb{E}^2$, we instead only take the rational numbers $\mathbb{Q}^2$. This is of course not equivalent to the set of real numbers; see the appendix for a proof. However it still poses quite interesting questions. In fact, we can give an explicit value for the chromatic number:

Theorem 11.1 (Woodall's Chromatic Number)**.**

$$\chi(\mathbb{Q}^2) = 2.$$

Proof. We will partition the set $\mathbb{Q}^2$ into subsets such that the difference between their respective coordinates has an odd denominator. In other words, for any two points $(r_1, r_2), (q_1, q_2) \in \mathbb{Q}^2$, (r_1, r_2) and (q_1, q_2) will be in the same subset if the denominators of $r_1 - q_1$ and $r_2 - q_2$ are both odd numbers upon simplification. The reason for this odd (pun intended) method is the following:

Lemma 11.2. *If the distance between (r_1, r_2) and (q_1, q_2) is 1, then they are in the same subset.*

Proof. Since the distance between (r_1, r_2) and (q_1, q_2) is 1, by the distance formula, we have $(r_1 - q_1)^2 + (r_2 - q_2)^2 = 1$. Now since $r_1, r_2, q_1, q_2 \in \mathbb{Q}$, we can let

$$r_1 - q_1 = \frac{a}{b}$$

and

$$r_2 - q_2 = \frac{c}{d}$$

for $a, b, c, d \in \mathbb{Z}$. Substituting, we get

$$(\frac{a}{b})^2 + (\frac{c}{d})^2 = 1 \Longrightarrow a^2d^2 + b^2c^2 = b^2d^2.$$

Then b and d must be both odd, so our above definition works. ∎

Now as we did with the other bounded cases in other problems, the coloring can be achieved by simply translating the base case. In each subset we color points of the form $(\frac{o}{o}, \frac{o}{o})$ and $(\frac{e}{o}, \frac{e}{o})$ red and points of the form $(\frac{o}{o}, \frac{e}{o})$ and $(\frac{e}{o}, \frac{o}{o})$ blue, where o represents an odd number and e represents an even number. This provides the unit distance coloring that we seek. ∎

Similar bounds have been found for the case of $\mathbb{Q}^n$, for some n, although there is no known general formula. The bounds and solutions for cases $n = 3$ to $n = 8$ are shown below.

n	$\chi(\mathbb{Q}^n)$
3	$= 2$
4	$= 4$
5	≥ 8
6	≥ 10
7	≥ 15
8	≥ 16

12. Conclusion

The Hadwiger-Nelson Problem is a quite interesting and mind-boggling problem. Originally a simple "fun" question, it has captivated the minds of mathematicians throughout the tweniteth century and beyond. It has connected seemingly unrelated fields of mathematics together in nontrivial ways, such as measure theory and topology. New discoveries are being made till date. We conclude with some conjectures, the proof of which are left to future readers.

Open Question 12.1 (Erd"os'). *Given S, find the S-chromatic number $\chi_S(\mathbb{E}^2)$ of the plane.*

Conjecture 12.2 (Total Chromatic Number). *For any graph G, $\chi_2(G) \leq \Delta(G) + 2$, where $\Delta(G)$ denotes the maximum degree of a vertex in G.*

Conjecture 12.3 (Hadwiger's). *Every connected n-chromatic graph G is contractible to K_n, where an edge contraction of a graph G consists of deleting an edge and attaching its incident vertices.*

Of course, quite a lot of money awaits you if you manage to improve bounds and show that $\chi(\mathbb{E}^2) = 7$! (Don't worry, the ! is an exclamation mark, not a factorial.)

13. Appendix

[1] [3] [4] [5] [6] [11] [9] [10] [7] [2] [8]

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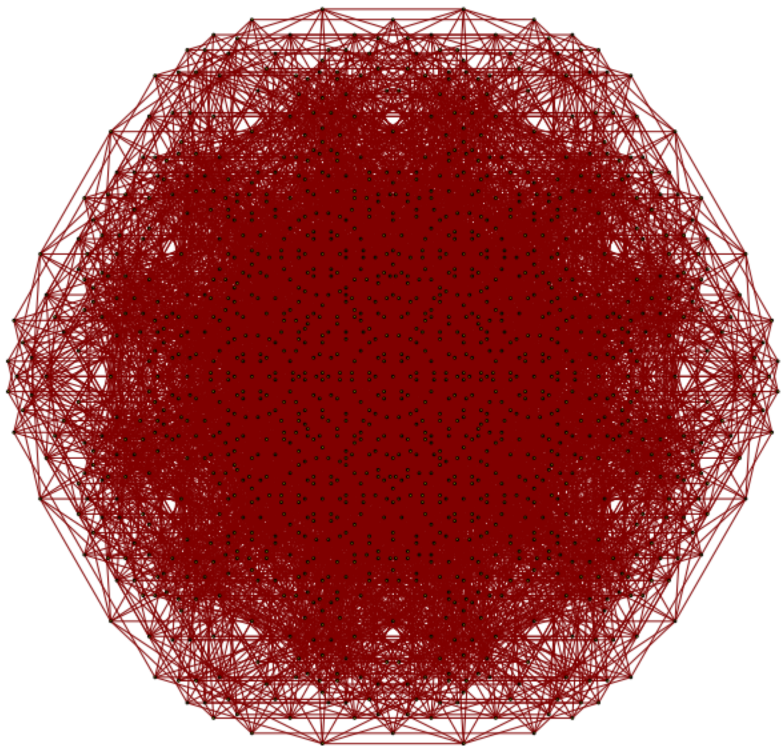

Figure 14. The monster.

12037 Brownwood Drive, Frisco, TX 75035
Email address: `samarthdas870@gmail.com`

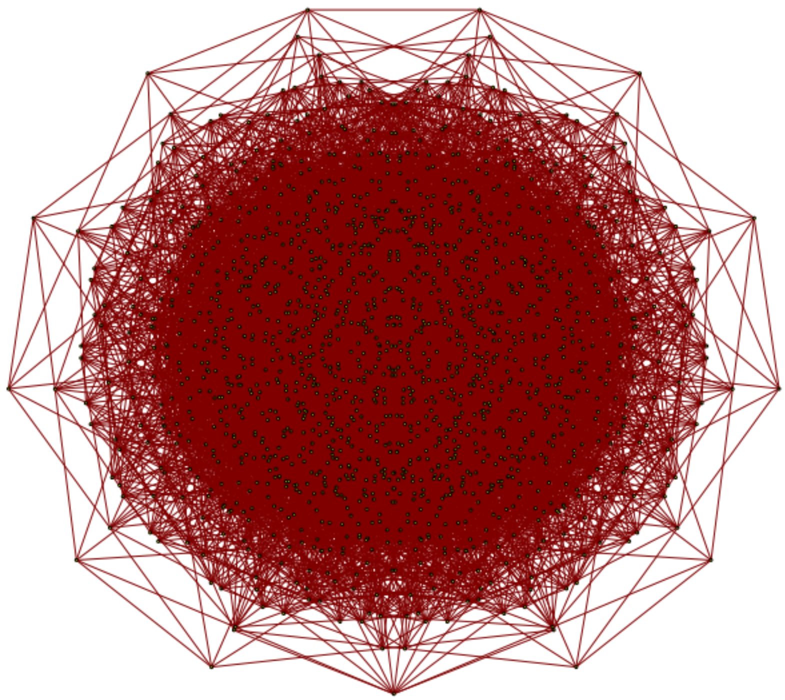

Figure 15. The Small Swarmonster.

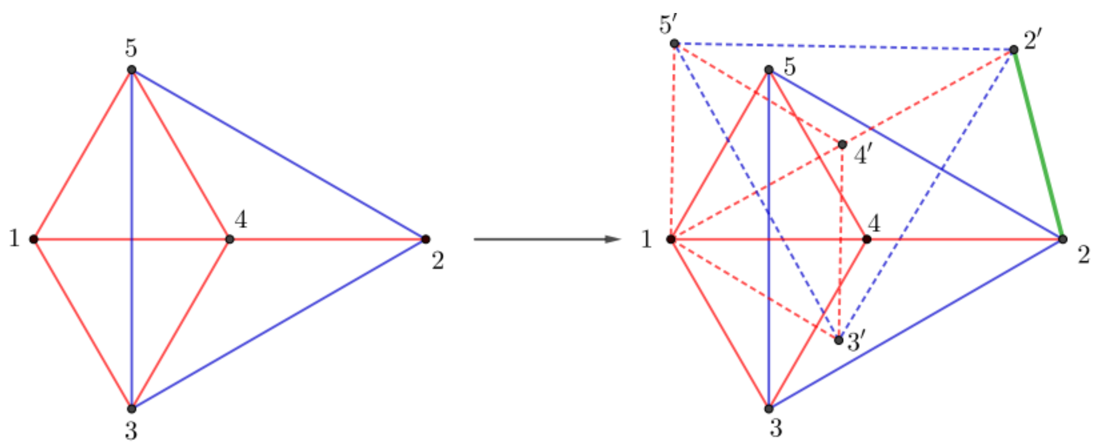

Figure 16. A Tesseractic tiling.

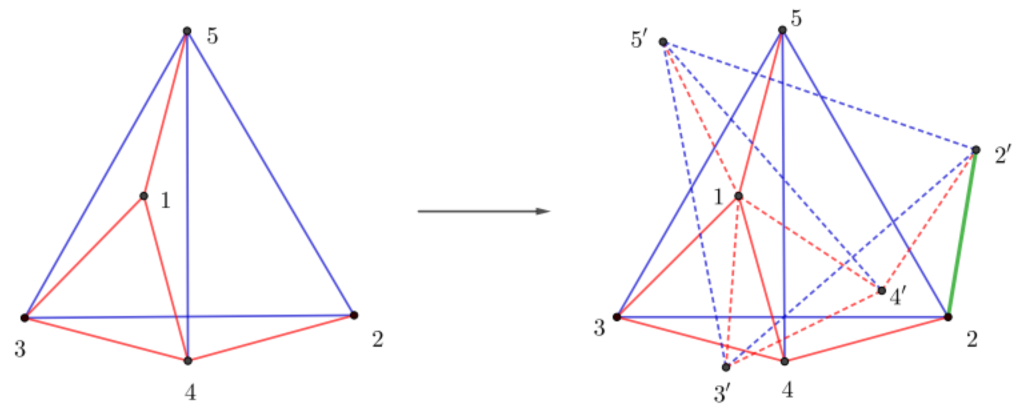

Figure 17. A Tesseractic tiling.

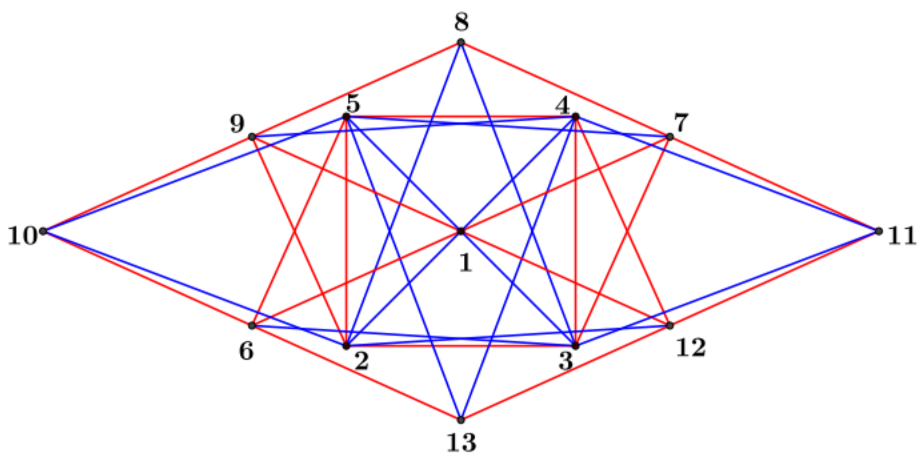

Figure 18. A Tesseractic tiling.

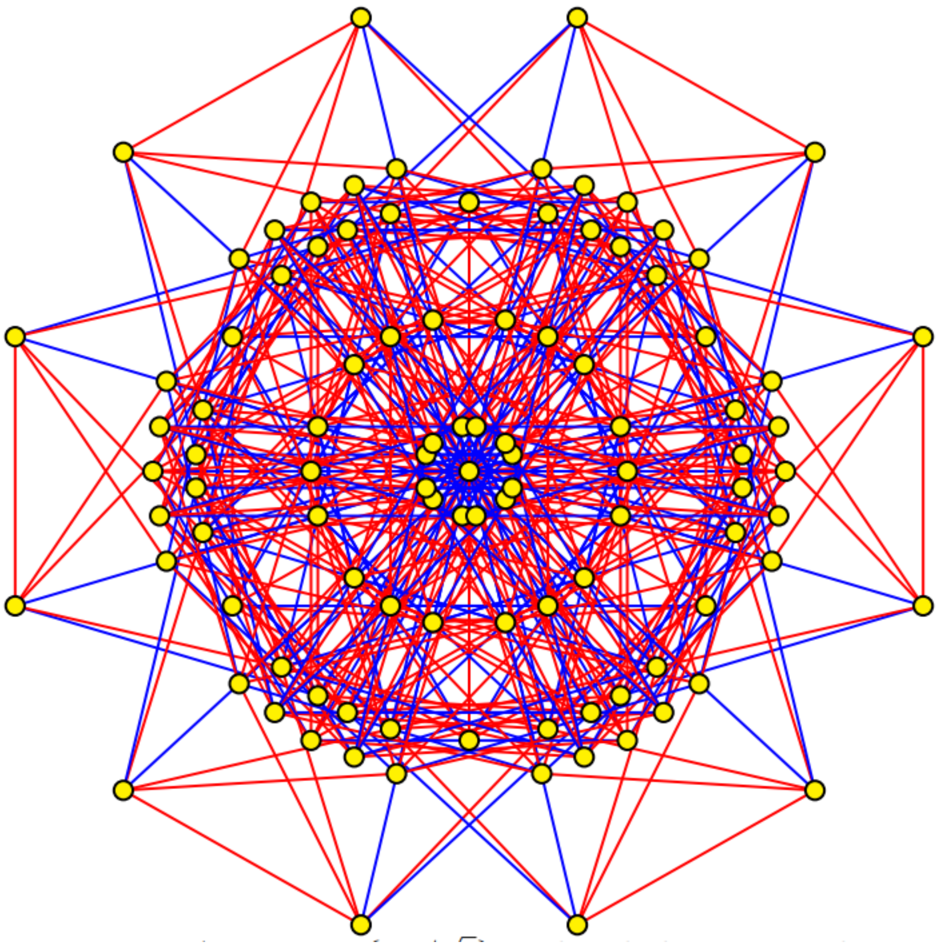

Figure 19. A Tesseractic tiling.

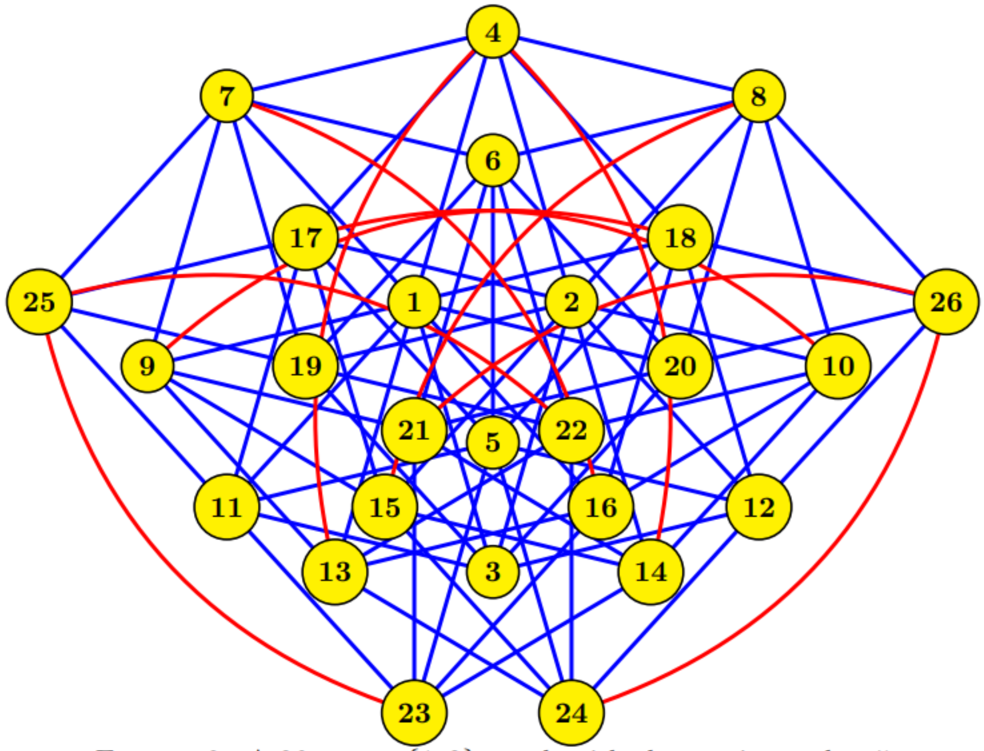

Figure 20. A Tesseractic tiling.

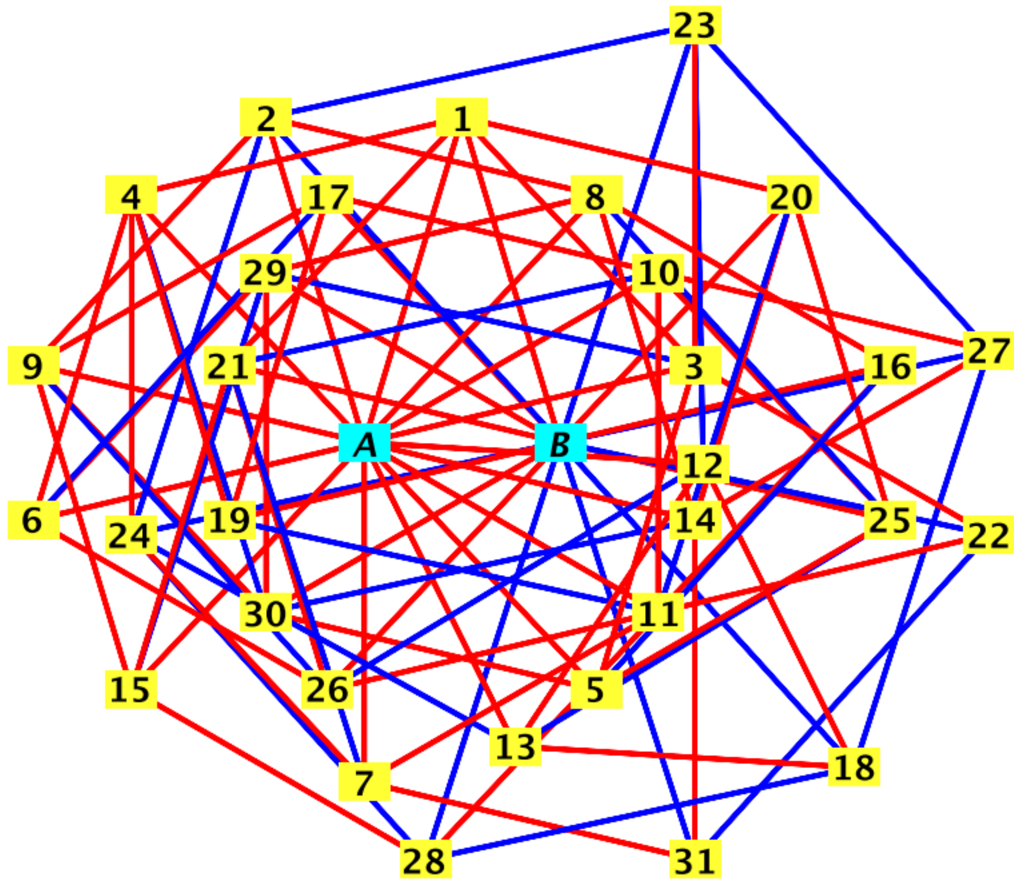

Figure 21. A Tesseractic tiling.

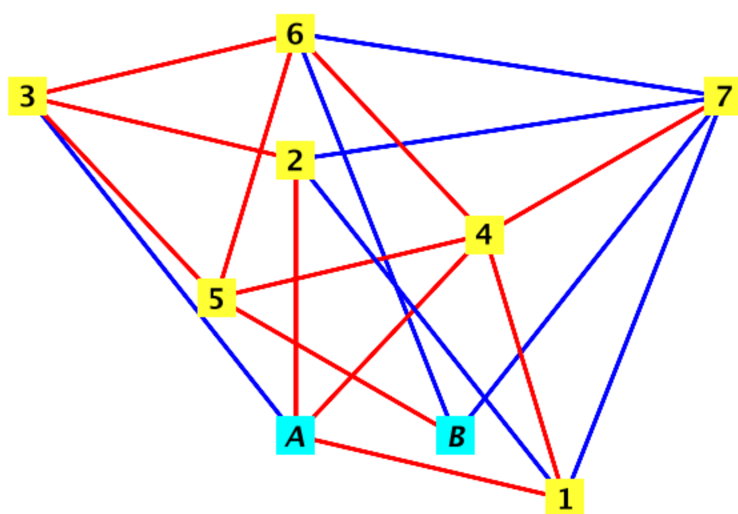

Figure 22. A Tesseractic tiling.

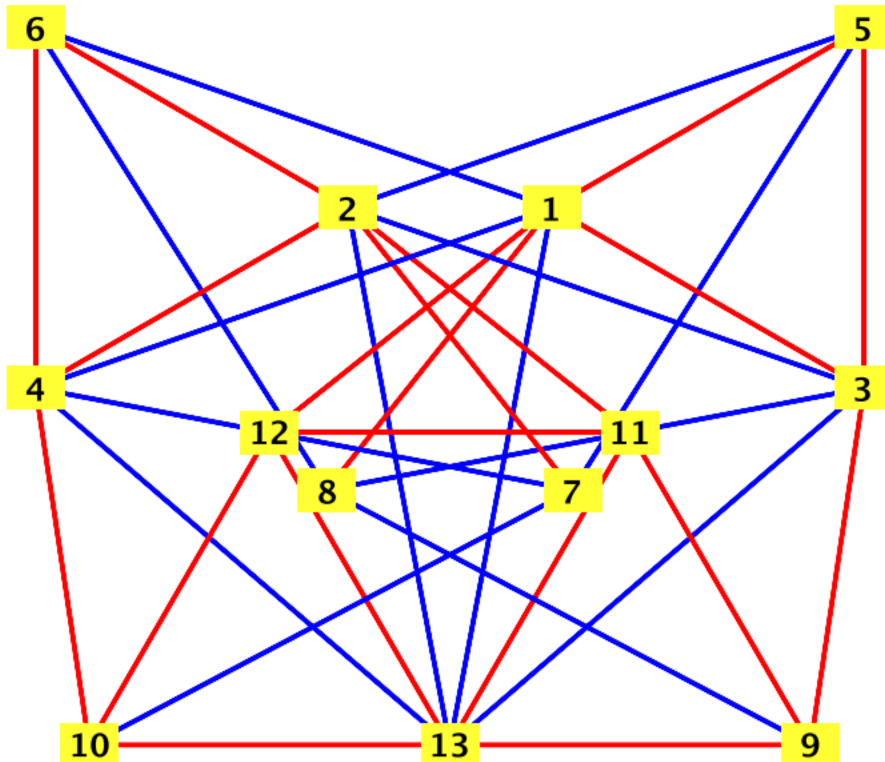

Figure 23. A Tesseractic tiling.