Ehrhart theory- On the Discrete Volume of Lattice Polytopes

Ritisha Bansal

July 17, 2023

Introduction to Ehrhart theory

The Ehrhart theory was devised by Eugene Ehrhart in 1962 who studied the relationship between an object's continuous volume and its discrete volume.

Introduction to Ehrhart theory

The Ehrhart theory was devised by Eugene Ehrhart in 1962 who studied the relationship between an object's continuous volume and its discrete volume.

This theory serves as a generalisation of Pick's theorem for polytopes in higher dimensions. Ehrhart studied how the number of lattice points inside an object changed as the object was scaled up in size.

Introduction to Ehrhart theory

The Ehrhart theory was devised by Eugene Ehrhart in 1962 who studied the relationship between an object's continuous volume and its discrete volume.

This theory serves as a generalisation of Pick's theorem for polytopes in higher dimensions. Ehrhart studied how the number of lattice points inside an object changed as the object was scaled up in size.

The Ehrhart theory has wide applications ranging from number theory, commutative algebra and enumerative combinatorics.

Pick's theorem

Theorem

(Pick's theorem). Given a convex integral polygon P, let the number of lattice points strictly interior to P be I and number of lattice points on the boundary of P be B. Then, the area of the polytope is-

$$A = I + \frac{B}{2} - 1.$$

Pick's theorem

Theorem

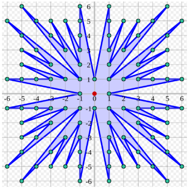
(Pick's theorem). Given a convex integral polygon P, let the number of lattice points strictly interior to P be I and number of lattice points on the boundary of P be B. Then, the area of the polytope is-

$$A = I + \frac{B}{2} - 1.$$

Example

The area of this convex polygon is,

$$A = 1 + \frac{96}{2} - 1 = 48.$$



Describing polytopes

There are two different ways to define convex polytopes are.

Describing polytopes

There are two different ways to define convex polytopes are.

1. Vertex description- Using the vertex description, a convex polytope $P \in \mathbb{R}^d$ is the convex hull of a finite set of points $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^d$. To be precise, polytope P is the smallest convex set containing those points. $P = conv\{v_1, v_2, \ldots, v_n\}$.

Describing polytopes

There are two different ways to define convex polytopes are.

1. Vertex description- Using the vertex description, a convex polytope $P \in \mathbb{R}^d$ is the convex hull of a finite set of points $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^d$. To be precise, polytope P is the smallest convex set containing those points. $P = conv\{v_1, v_2, \ldots, v_n\}$.

2. Hyperplane description-By the hyperplane description, a convex polytope P is the bounded intersection of finitely many half-spaces defined by linear inequalities.

Lattice-point enumeration

Ehrhart theory deals with computing the discrete volume of a polytope by counting the number of its integer lattice points which is the lattice-point enumerator.

Lattice-point enumeration

Ehrhart theory deals with computing the discrete volume of a polytope by counting the number of its integer lattice points which is the lattice-point enumerator.

Definition

For a positive integer t, the t^{th} dilate of $P \in \mathbb{R}^d$ is tP, and

$$tP = \{(tx_1, tx_2, \cdots, tx_d) : (x_1, x_2, \cdots, x_d) \in P\}.$$

Lattice-point enumeration

Ehrhart theory deals with computing the discrete volume of a polytope by counting the number of its integer lattice points which is the lattice-point enumerator.

Definition

For a positive integer t, the t^{th} dilate of $P \in \mathbb{R}^d$ is tP, and

$$tP = \{(tx_1, tx_2, \cdots, tx_d) : (x_1, x_2, \cdots, x_d) \in P\}.$$

Definition

The lattice point enumerator of $P \in \mathbb{R}^d$ when evaluated at t is,

$$L_p(t) = |tP \cap \mathbb{Z}^d|.$$

The value of $L_p(t)$ is the discrete volume of tP.

Ehrhart series

Definition

The Ehrhart series is another important tool for analyzing a polytope P. It is the generating function of the lattice point enumerator of P and can be defined as,

$$Ehr_p(z) = \sum_{t>=0} L_p(t) z^t.$$

Example

The Ehrhart series of the origin is-

$$\frac{1}{(1-z)} = 1 + z + z^2 + z^3 + \cdots$$

Unit D-Cube

The unit d-cube $\square$ is a polytope whose vertices are are all of the points in $\mathbb{R}^d$ such that every coordinate is either 0 or 1:

$$\square = conv\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ } or \text{ } 1 \text{ } for \text{ } 1 <= i <= d\}.$$

$$= \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 <= x_i <= 1 \text{ } for \text{ } 1 <= i <= d\}.$$

Unit D-Cube

The unit d-cube $\square$ is a polytope whose vertices are are all of the points in $\mathbb{R}^d$ such that every coordinate is either 0 or 1:

$$\square = conv\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1 \text{ for } 1 <= i <= d\}.$$
$$= \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 <= x_i <= 1 \text{ for } 1 <= i <= d\}.$$

Theorem

The lattice-point enumerator of $\square$ *is,*

$$L_\square(t) = (t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k$$

Example

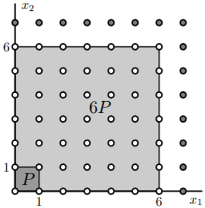


Figure: 6th dilate of unit-cube

Example

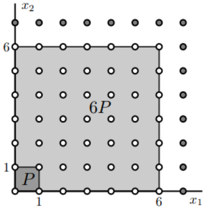


Figure: 6th dilate of unit-cube

Computing the Ehrhart series of the unit d-cube-

$$Ehr_{\square} = \sum_{t>=0} (t+1)^d z^t$$
$$= \sum_{k=1}^{d} \frac{A(d,k) z^k}{(1-z)^{d+1}}.$$

Standard Simplex

The standard simplex denoted by $\triangle$ in dimension d is the convex hull of $(d+1)$ points $e_1, e_2, \cdots, e_d$ and the origin. Here is e_j is unit vector with a 1 in the j^{th} position while the rest are 0 vectors.

$$\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leq 1 \text{ and all } x_k \geq 0\}.$$

The t dilate of the standard simplex is given by,

$$t\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d <= t \text{ and all } x_k >= 0\}.$$

Standard Simplex

The standard simplex denoted by $\triangle$ in dimension d is the convex hull of $(d+1)$ points $e_1, e_2, \cdots, e_d$ and the origin. Here is e_j is unit vector with a 1 in the j^{th} position while the rest are 0 vectors.

$$\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leq 1 \text{ and all } x_k \geq 0\}.$$

The t dilate of the standard simplex is given by,

$$t\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d <= t \text{ and all } x_k >= 0\}.$$

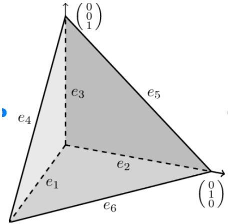


Figure: 3-D standard simplex

To compute the discrete volume of $\triangle$ we use a counting function.

$$m_1 + m_2 + \cdots + m_d \leq t.$$

$$m_1 + m_2 + \cdots + m_{d+1} = t.$$

To compute the discrete volume of $\triangle$ we use a counting function.

$$m_1 + m_2 + \cdots + m_d \leq t.$$

$$m_1 + m_2 + \cdots + m_{d+1} = t.$$

The counting function is,

$$t\triangle \in \mathbb{Z}^d = \text{const}\left(\left(\sum_{m_1>=0} z^{m_1}\right)\left(\sum_{m_2>=0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1}>=0} z^{m_{d+1}}\right) z^{-t}\right).$$

$$= \text{const}\left(\frac{1}{(1-z)^{d+1} z^t}\right).$$

To compute the discrete volume of $\triangle$ we use a counting function.

$$m_1 + m_2 + \cdots + m_d \le t.$$

$$m_1 + m_2 + \cdots + m_{d+1} = t.$$

The counting function is,

$$t\triangle \in \mathbb{Z}^d = \text{const}\left(\left(\sum_{m_1>=0} z^{m_1}\right)\left(\sum_{m_2>=0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1}>=0} z^{m_{d+1}}\right)z^{-t}\right).$$
$$= \text{const}(\frac{1}{(1-z)^{d+1}z^t}).$$

Thus, $Ehr_{\triangle}(z) = \frac{1}{(1-z)^{d+1}}$. Using the binomial series we get,

$$\frac{1}{(1-z)^{d+1}} = \sum_{k\ge 0}\binom{d}{k}z^k.$$

Thus, $L_{\triangle}(t) = \binom{d+t}{d}$

Cross-polytopes

The hyperplane description of a cross polytope $\diamond\ \mathbb{R}^d$ hyperplane description is-

$$:= \{(x_1, x_2, \ldots, x_d) \in R^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}.$$

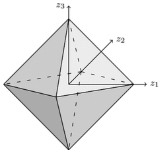


Figure: 3-D cross polytope- $BiPyr(Q)$

Cross-polytopes

The hyperplane description of a cross polytope $\diamond$ $\mathbb{R}^d$ hyperplane description is-

$$:= \{(x_1, x_2, \ldots, x_d) \in R^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}.$$

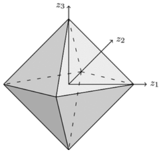


Figure: 3-D cross polytope- $BiPyr(Q)$

A d dimensional cross-polytope can be defined as the bipyramid over a $(d-1)$ dimensional cross polytope Q with vertices $v_1, v_2, \ldots, v_m$ such that Q contains the origin. We define $BiPyr(Q)$ as-

$$conv\{(v_1, 0), (v_2, 0), \ldots, (v_m, 0), (0, \ldots, 0, 1) \text{ and } (0, \ldots, 0, -1)$$

$$L_{BiPyr(Q)}(t) = 2 + 2L_Q(1) + 2L_Q(2) + \cdots + 2L_Q(t-1) + L_Q(t).$$

Theorem If Q contains the origin, then $Ehr_{BiPyr(Q)}(z) = \frac{1+z}{1-z} Ehr_Q(z)$. The cross poltyope in dimension 0 is the origin with Ehrhart series $\frac{1}{(1-z)}$. Thus, the higher dimensional cross polytopes can be computed recursively by the formula-

$$Ehr_{\diamond}(z) = \frac{(1+z)^d}{(1-z)^{d+1}}.$$

Ehrhart theory

Theorem

Ehrhart's theorem- *Given a convex integral polytope $P \in \mathbb{R}^d$, the lattice-point enumerator $L_P(t)$ of P is a rational polynomial in t of degree d which we call the Ehrhart polynomial.*

$$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

Ehrhart Macdonald Reciprocity

Theorem

Ehrhart Macdonald Reciprocity- *Given a convex polytope* $P \in \mathbb{R}^d$*, evaluating* $L_P(t)$ *at negative integers yields,*

$$L_P(-t) = (-1)^d L_{P^\circ}(t).$$

From Discrete to Continuous Volume

Theorem

For a given convex polytope $P \in \mathbb{R}^d$ let its Ehrhart polynomial be,

$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$. Then c_d equals the volume of P

.

From Discrete to Continuous Volume

Theorem

For a given convex polytope $P \in \mathbb{R}^d$ let its Ehrhart polynomial be,

$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$. *Then c_d equals the volume of P*

. Proof

$$volP := \lim_{t\to\infty} \frac{1}{t^d} |P \cap \frac{1}{t} Z^d| = \lim_{t\to\infty} \frac{1}{t^d} |tP \cap Z^d|.$$
$$= \lim_{t\to\infty} \frac{1}{t^d} L_P(t).$$

We now have,

$$volP = \lim_{t\to\infty} \frac{c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + 1}{t^d}.$$
$$= \lim_{t\to\infty} (c_d + c_{d-1} t^{-1} + \cdots + c_1 t^{-d+1} + t^{-d}).$$
$$= c_d.$$

Decoding coefficients

Decoding the Second Leading coefficient-

Theorem

Suppose $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$ is the Ehrhart polynomial of an integral polytope P. Then,

$$c_{d-1} = \frac{1}{2} \sum_{F \text{ } facetofP} vol(F).$$

Decoding coefficients

Decoding the Second Leading coefficient-

Theorem

Suppose $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$ is the Ehrhart polynomial of an integral polytope P. Then,

$$c_{d-1} = \frac{1}{2} \sum_{F \ facetofP} vol(F).$$

Decoding the Last coefficent-
The constant term c_0 of the Ehrhart polynomial is the Euler characteristic of P and is equal to 1.

Ehrhart positivity

We conclude with an open field of research in Ehrhart theory-Ehrhart positivity. A convex integral polytope P is said to have Ehrhart positivity if $L_P(t)$ has all positive coefficients. This leads us the central question of this field of research.

Open Question

Which faimilies of integeral polytopes have Ehrhart positivity?