DISCRETE VOLUME OF LATTICE POLYTOPES

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Abstract- In this paper, we will overview the fundamentals of the Ehrhart theory, specifically Ehrhart polynomials and their associated Ehrhart series. The Ehrhart theory was proposed by Eugene Ehrhart in 1962 who at the time was as a mathematics teacher at a lycées. This theory shows that the number of lattice points contained by the t^{th} dilate of a d-dimensional polytope is a rational polynomial in t of degree d called the Ehrhart polynomial. We will examine the Ehrhart polynomials of some common polytopes and properties of these polynomials, a proof of the Ehrhart theory and an open problem in this field.

1. Introduction

Since, Ehrhart's original works in 1960, Ehrhart theory has developed as a key topic at the intersection of polyhedral geometry, number theory, commutative algebra and enumerative combinatorics. Its diverse applications are owing to the fact that it connects the worlds of discrete and continuous mathematics in a fascinating way. Ehrhart studied the relationship between an object's continuous volume which is its normal or intuitive volume and its discrete volume-a different sense of volume given by the lattice points contained inside it. In the mathematics of lattice points, the first well-known result is Pick's theorem which gives the area of a convex polygon in terms of lattice points inside it and on its boundary. Ehrhart devised a different approach to serve as a generalisation for polytopes with higher dimensions. Ehrhart instead studied how the number of lattice points inside an object changed as the object was scaled up in size. For this purpose, he defined the lattice-point enumerator function $L_P(t)$ in t of an object, which counted the number of lattice points in the object after being scaled up by a factor of t for positive integers t. The central theorem of Ehrhart theory is theorem 10.1 which says,

$$L_P(t) = c_d t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + c_0.$$

Indeed, the lattice-point enumerator of a polytope P is an Ehrhart polynomial. Connecting back to Pick's theorem and volume, Ehrhart discovered that the leading coefficient of a polytope's lattice-point enumerator always equaled the polytope's volume. Moreover, the associated generating function for the lattice-point enumerator of P called the Ehrhart series can be defined as,

$$Ehr_P(z) = 1 + \textstyle\sum_{t\geq 1} L_P(t) z^t.$$

The structure of the paper is as follows. We start by an in-depth look at Pick's theorem in section 2 providing a traditional proof for it. Section 3 establishes fundamentals of polytopes and we then move on to laying out the basics of lattice-point enumeration in section 4 with sections 5,6,7 and 8 computing the lattice-point enumerators of several common polytopes. Continuing in section 9, we will explore the role of generating functions in Ehrhart theory and then go on to introduce cones and integer-point transform in section 10. The main results of Ehrhart theory are provided in section 11.Sections 12 and 13 deal with interpreting

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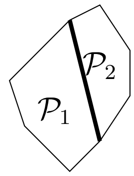

Figure 1. decomposition of convex polygon

coefficients and interpolation. Section 14 briefly explores Ehrhart quasipolynomials and lastly in section 14 we explore an active area of research in this field.

2. Pick's theorem

Pick's theorem is the most famous theorem involving lattice-point enumeration and it is fundamental to this paper as it provided the basis for developing Ehrhart theory. The theorem named in honour of its discover Georg Alexander Pick (1859-1942) presents the surprising fact that the area A of polynomial P can be computed simply by counting lattice points. The theory of Ehrhart polynomials can be seen as a higher-dimensional generalization of Pick's theorem in the Euclidean plane.

Theorem 2.1. *(Pick's theorem). Given a convex integral polygon P, let the number of lattice points strictly interior to P be I, and let the number of lattice points on the boundary of P be B. Then, the area of the polytope is-*

$$A = I + \tfrac{B}{2} - 1.$$

Lemma 2.2. *Pick's identity has an additive character: we can always decompose P into the union of two integral polygons P_1 and P_2 by joining two vertices of P with a line segment as shown by figure 1.The validity of Pick's identity for P follows from the validity of the Pick's identity for P_1 and P_2.*

Denote the area, number of interior lattice points, and number of boundary lattice points of P_k by A_k, I_k, and B_K respectively, for $k = 1, 2$. Clearly,

$$A = A_1 + A_2.$$

Furthermore, if we denote the number of lattice points on the edge common to P_1 and P_2 by L, then

$$I = I_1 + I_2 + L - 2 \quad \text{and} \quad B = B_1 + B_2 - 2L + 2.$$

Thus

$$I + \frac{B}{2} - 1 = I_1 + I_2 + L - 2 + \frac{B_1}{2} + \frac{B_2}{2} - L + 1 - 1.$$
$$= I_1 + \frac{B_1}{2} - 1 + I_2 + \frac{B_2}{2} - 1.$$

This proves the claim. Note that our proof also shows that the validity of Pick's identity for P_1 follows from the validity of Pick's identity for P and P_2. Now, every convex polygon can be decomposed into triangles that share a common vertex. Hence it suffices to prove Pick's theorem for triangles.

Lemma 2.3. *Pick's theorem holds for all integral rectangles R whose sides are parallel to the axes.*

Proof. Let the width of R be w and the height of R be h. Then, without loss of generality let the bottom-left vertex of R be (0, 0) and the top-right vertex of R be (w, h). So we have,

$$A = wh. \tag{2.1}$$

The lattice points strictly inside R form a rectangular grid with bottom-left vertex (1, 1) and top-right vertex $(w-1, h-1)$, so we have,

$$I = (w-1)(h-1). \tag{2.2}$$

The horizontal edges of R each have length w, so they contain $w+1$ lattice points each. Likewise, the vertical edges each have length h, so they contain $h+1$ lattice points each. This gives that,

$$B = 2(w+1) + 2(h+1) - 4 = 2w + 2h. \tag{2.3}$$

Putting together (2.1), (2.2), and (2.3) we have,

$$\begin{aligned} I + \frac{B}{2} - 1 &= (w-1)(h-1) + \frac{2w+2h}{2} - 1. \\ &= wh - w - h + 1 + w + h - 1 \\ &= wh = A. \end{aligned}$$

■

Lemma 2.4. *Pick's theorem holds for all integral right triangles T whose legs are parallel to the axes.*

Proof. Let the legs of T have lengths w and h. Then, without loss of generality let the vertices of T be (0, 0), (w, h), and $(w, 0)$. Thus, we have,

$$A = \frac{wh}{2}. \tag{2.4}$$

Let H be the number of lattice points on the hypotenuse of T .Then, if we use the same definition of rectangle, we see that the number of internal lattice points of T is half the number of internal lattice points of R that do not lie on the hypotenuse of T . Thus, with 2 accounting for vertices,

$$I = \frac{(w-1)(h-1) - H + 2}{2}. \tag{2.5}$$

For the boundary, the horizontal and vertical legs contain $w+1$ and $h+1$ lattice points, respectively. The hypotenuse contains H lattice points. Subtracting 3 to account for the vertices gives,

$$B == (w+1) + (h+1) + H - 3 = w + h + H - 1. \tag{2.6}$$

Putting together (2.4), (2.5), and (2.6) we have,

$$\begin{aligned} I+\frac{B}{2}-1 &= \frac{(w-1)(h-1)-H+2}{2}+\frac{w+h+H-1}{2}-1. \\ &= \frac{wh-w-h+1+w+h-1}{2}. \\ &= \frac{wh}{2}=A. \end{aligned}$$

■

Thus, with the above three lemmas, we can piece together the proof of Pick's theorem.

We now go on to derive the general form of the lattice-point enumerator for all convex polytopes using Pick's theorem.

Theorem 2.5. *Given a convex integral polygon P, let its area be A and let the number of lattice points on its boundary be B. Then,*

$$L_p(t)=At^2+\tfrac{B}{2}t+1.$$

Proof. Consider tP, the scaling of P to its t dilate. Let its area be A_t, the number of lattice points in its interior be I_t and those on its boundary be B_t. We have,

$$L_p(t)=I_t+B_t. \tag{2.7}$$

Since tP is P under a dilation by a factor of t, we have that the area increases by a factor of t^2. This gives,

$$A_t=At^2. \tag{2.8}$$

Thus to finish, Pick's theorem combined with (2.7) and (2.8) gives,

$$\begin{aligned} At &= I_t+\frac{B_t}{2}-1. \\ At &= I_t+B_t-\frac{B_t}{2}-1. \\ At &= L_p(t)-\frac{B_t}{2}-1. \\ L_p(t) &= A_t+\frac{B_t}{2}+1. \\ L_p(t) &= At^2+\frac{B}{2}t+1. \end{aligned}$$

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3. Description of Polytopes

In this paper, we will only discuss convex polytopes. The two different ways to define convex polytopes are the vertex description and the hyperplane description.

Definition 3.1. Using the vertex description, a convex polytope $P \in \mathbb{R}^d$ is the convex hull of a finite set of points $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^d$. You cane imagine this by film wrapping around the vertices of a polytope, then the region enclosed by the film is the polytope's convex hull. We denote the convex hull of P by $P=\text{conv}\{v_1, v_2, \ldots, v_n\}$.To be precise polytope P is the smallest convex set containing those points; that is,

$$P = \{\lambda_1 v_1 + \lambda_2 v_2 \cdots + \lambda_n v_n \ : all\ \lambda_k >= 0\ and\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

Definition 3.2. If we instead use the hyperplane description, a convex polytope $P \in \mathbb{R}^d$ is the bounded intersection of finitely many d-dimensional half spaces and $(d-1)$-dimensional hyperplanes. A hyperplane $H \in \mathbb{R}^d$ is a $(d-1)$ dimensional subspace of a d-dimensional space. A half-space $\mathcal{H} \subset \mathbb{R}^d$ is the part of a d-dimensional space that lies on a given side of a $(d-1)$-dimensional hyperplane. We call a hyperplane H a supporting hyperplane of a polytope P if P is completely contained in one of the two half spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ bounded by H such that $P \in \mathcal{H}_1$ or $P \in \mathcal{H}_2$. For some, some $a \in \mathbb{R}^d$ and some constant b.

$$\mathcal{H} := \{x \in \mathbb{R} : a.x \geq b\} \text{ or } \mathcal{H} := \{x \in \mathbb{R} : a.x \leq b\}.$$

The fact that every polytope has both vertex and hyperplane description which are equivalent is highly nontrivial both conceptually and algorithmically.

Definition 3.3. The dimension of the P is the dimension of the affine space spanned by P.

$$\text{span } P := \{x + \lambda(y - x) : x, y \in P, \lambda \in \mathbb{R}\}.$$

The face of a d-dimensional polytope can have any dimensionality less than or equal to d. In particular, the $(d-1)$-dimensional faces are called facets, the 1-dimensional (line segment) faces are called edges, and the 0-dimensional (point) faces are called vertices. A convex d-polytope has at least $(d+1)$ vertices. A convex d-polytope with exactly $(d+1)$ vertices is called a d-simplex. Also, note that Ehrhart theory is concerned with polytopes with integer or rational vertices i.e a rational polytope.

4. Lattice Point Enumeration

Lattice point enumerators-Ehrhart theory deals with computing the discrete volume of a polytope by counting the number of its integer lattice points. We are interested in how the number of lattice points differs as P is scaled up.

Definition 4.1. For a positive integer t, the t^{th} dilate of $P \in \mathbb{R}^d$ is tP, and

$$tP = \{(tx_1, tx_2, \cdots, tx_d) : (x_1, x_2, \cdots, x_d) \in P\}.$$

Definition 4.2. The lattice point enumerator of $P \in \mathbb{R}^d$, which counts lattice points inside tP when evaluated at t is,

$$L_p(t) = |tP \cap \mathbb{Z}^d|.$$

We can think of this in another way as leaving P fixed and shrinking the integer lattice that is,

$$L_p(t) = |P \cap \tfrac{1}{t}\mathbb{Z}^d|.$$

The value of $L_p(t)$ is called the discrete volume of tP.

Definition 4.3. The Ehrhart series is another important tool for analyzing a polytope P. It is the generating function of the lattice point enumerator of P and can be defined as,

$$Ehr_p(z) = 1 + \sum_{t \geq 1} L_p(t) z^t$$

$$Ehr_p(z) = \sum_{t>=0} L_p(t) z^t.$$

Note -The generating function for $x_1 + x_2 + \cdots + x_{d+1}$ is,

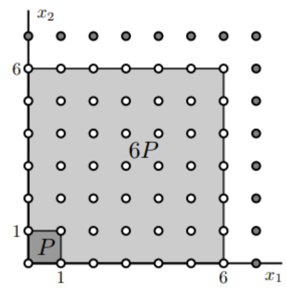

Figure 2. 6th dilate of unit-cube

$$(1 + z + z^2 + \cdots)^{d+1} = \frac{1}{(1-z)^{d+1}}.$$

The coefficient of z^k in this generating function, when expanded fully, equals the number of ways to choose non-negative integers $x_1, x_2, \ldots, x_{d+1}$ such that $x_1 + x_2 + \cdots + x_{d+1} = k$. We will be using this generating function for lattice-point enumeration of a standard simplex.

5. Unit D-Cube

We first begin with lattice point enumeration of unit d-cube:=$[0,1]^d$ which is a generalisation of the 2-D unit square and 3-D unit cube. Denoted by $\square$, it is the is the polytope whose vertices are all of the points in $\mathbb{R}^d$ such that every coordinate is either 0 or 1:

$$\square = conv\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \ or \ 1 \ for \ 1 <= i <= d\}.$$
$$= \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 <= x_i <= 1 \ for \ 1 <= i <= d\}.$$

The interior of the unit d-cube is,

$$= \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : 0 < x_i < 1 \ for \ 1 <= i <= d\}.$$

Theorem 5.1. *The lattice-point enumerator of unit-d cube is,*

$$L_{\square}(t) = (t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k$$

and the lattice-point enumerator of the interior unit-d cube is,

$$= (t-1)^d.$$

Proof. A given lattice point $(x_1, x_2, \cdots, x_d)$ can have $x_i = 0, 1, \cdots, t$. Thus, since each of the d coordinates had $t+1$ possible values, $L_{\square}(t) = (t+1)^d$. ∎

The Ehrhart series of the unit-d cube takes a special form as it can be expressed in terms of the Eulerian number $A(d,k)$

Definition 5.2. The Euelerian number $A(d,k)$ can be defined as the number of ways to arrange numbers from 0 to d such that k numbers are greater than the previous number. The generating function for $A(d,k)$ is,

$$\sum_{j>=0} j^d z^j = \sum_{k=0}^{d} \frac{A(d,k) z^k}{(1-z)^(d+1)}.$$

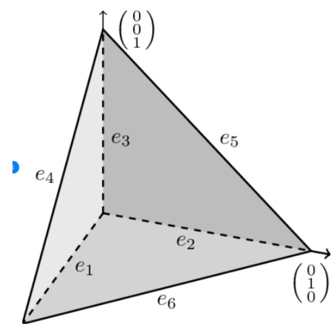

Figure 3. 3-D standard simplex

We can now express Ehrhart series of $\square$ in terms of Eulerian numbers:

$$\begin{aligned} Ehr_{\square} &= 1 + \sum_{t>=1} (t+1)^d z^t = \sum_{t>=0} (t+1)^d z^t. \\ &= \frac{1}{z} \sum_{t>=1} t^d z^t. \\ &= \sum_{k=1}^{d} \frac{A(d,k) z^k}{(1-z)^{d+1}}. \end{aligned}$$

The Eulerian numbers have many fascinating properties like $A(d,k) = A(d, d+1-k)$ which provide different ways of computing Ehrhart series of $\square$.

6. The Standard Simplex

Definition 6.1. The standard simplex denoted by $\triangle$ in dimension d is the convex hull of $(d+1)$ points $e_1, e_2, \cdots, e_d$ and the origin. Here is e_j is unit vector with a 1 in the j^{th} position while the rest are 0 vectors.

$$\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leq 1 \text{ and all } x_k \geq 0\}.$$

The t dilate of the standard simplex is given by,

$$t\triangle = \{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d <= t \text{ and all } x_k >= 0\}.$$

To compute the discrete volume of $\triangle$ we use a counting function as we are trying to count all integer solutions $\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ and $(m_1, m_2, \ldots, m_d) \geq 0\}$ to,

$$m_1 + m_2 + \cdots + m_d \leq t.$$

To translate this inequality in d variables to an equality $d+1$ variables, we introduce a slack variable m_{d+1}, such that $m_{d+1} \in \mathbb{Z}^d$ and $m_{d+1} \geq 0$. Thus,

$$m_1 + m_2 + \cdots + m_{d+1} = t.$$

The counting function is,

$$\begin{aligned} t\triangle \in \mathbb{Z}^d &= \text{const}\left(\left(\sum_{m_1>=0} z^{m_1}\right)\left(\sum_{m_2>=0} z^{m_2}\right)\cdots\left(\sum_{m_{d+1}>=0} z^{m_{d+1}}\right) z^{-t}\right). \\ &= \text{const}\left(\frac{1}{(1-z)^{d+1} z^t}\right). \end{aligned}$$

Thus, we have proved that the Ehrhart series of $\triangle$ is $Ehr_{\triangle}(z) = \frac{1}{(1-z)^{d+1}}$.
Using the binomial series we get,

$$\frac{1}{(1-z)^{d+1}} = \sum_{k\geq 0} \binom{d}{k} z^k.$$

Thus, we have proved that the lattice point enumerator of $\triangle$ is the polynomial $L_{\triangle}(t) = \binom{d+t}{d}$.
Incidentally $L_{\triangle}(t)$ has an alternative life in traditional combinatorics as shown below.

Definition 6.2. A Sterling number denoted by $s(n.k)$ is the number of ways to partition a set of n elements into k non-empty sets. The uniqueness of these numbers is that they are the first of the kind to be denoted as the coefficients in expansions of falling and rising factorials.

$L_{\triangle}(t)$ can be expressed in terms of the Sterling numbers in a unique way as,

$$\triangle = \frac{1}{d!} \sum_{k=0}^{d} (-1)^{d-k} stirl(d+1, k+1) t^k.$$

7. Bernoulli polynomials as Lattice Point Enumerators of Pyramids

We will explore the fascinating connection between Bernoulli polynomials and certain pyramids over unit cubes.

Definition 7.1. The **Bernoulli polynomials** named after Jacob Bernoulli, combine Bernoulli numbers and Binomial coefficients and are useful for series expansion of functions.The Bernoulli polynomials B_n can be defined by the generating function,

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k\geq 0} \frac{B_k(x)}{k!} z^k.$$

The **Bernoulli numbers** are $B_k := B_k(0)$ and have generating function,

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k\geq 0} \frac{B_k}{k!} z^k.$$

Theorem 7.2. *For integers $d >= 1$ and $n >= 2$,*

$$\sum_{k=0}^{n-1} k^d - 1 = \frac{1}{d}(B_d(n) - B_d).$$

This can be proved using the generating functions defined above.

For a polytope $Q \in \mathbb{R}^{d-1}$ with vertices $v_1, v_2, \ldots, v_m$ we define $Pyr(Q)$ as the convex hull of $(v_1, 0), (v_2, 0), \ldots, v_m, (0, 0, \ldots, 1)$. Given below is an example of the unit-cube.

A $(d-1)$ dimensional unit cube when embedded in $\mathbb{R}^d$ forms a d dimensional pyramid by adjoining one more vertex at $(0, 0, \ldots, 1)$. This geometric object has the given hyperplane description,

$$P = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : 0 <= x_1, x_2, \ldots, x_{d-1} \leq 1 - x_d \leq 1\}.$$

Thus, pyramid P is contained in the unit cube and its vertices are a subset of the vertices of the cube.

Lattice-point enumeration in integer dilates of P. The number of lattice points inside P is given by,

$$\{(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : 0 \leq m_k \leq t - m_d \leq t \ for \ k = 1, 2, \ldots, d-1\}.$$

When counting the solutions to $0 \leq m_k \leq t - m_d \leq t$, once we have chosen m_d from 0 to t, we have $t - m_d + 1$ independent choices for each of the integers $m_1, m_2, \ldots, m_{d-1}$. The lattice

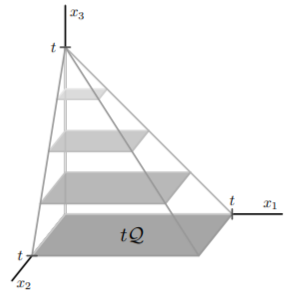

Figure 4. Lattice point enumeration of $tPyr(Q)$

point enumerator of P is the total number of solutions to the counting function. Thus,

$$L_p(t) = \sum_{m_d=0}^{t} (t - m_d + 1)^{d-1} = \sum_{k=1}^{t+1} k^{d-1}.$$
$$= \frac{1}{d}(B_d(t+2) - B_d).$$

Theorem 7.3. *the number of integer lattice points in $tPyr(Q)$ is by construction,*

$$L_{Pyr(Q)}(t) = 1 + L_Q(1) + L_Q(2) + \cdots + L_Q(t).$$
$$= 1 + \sum_{j=1}^{t} L_Q(j).$$

Proof. This is because there is one lattice point in which x_d coordinate is t which is a vertice, $L_Q(1)$ lattice points with x_d coordinate as $(t-1)$, $L_Q(2)$ lattice points with x_d coordinate as $(t-2)$ up to $L_Q(t)$ lattice points with $x_d = 0$ as these are the lattice points for the cube. ■

Theorem 6.3 allows us to compute the Ehrhart series of $Pyr(Q)$ from the Ehrhart series of Q:

Theorem 7.4. $Ehr_{Pyr(Q)}(z) = \frac{Ehr_Q(z)}{1-z}$

Proof.

$$Ehr_{Pyr(Q)}(z) = 1 + \sum_{t\geq 1} L_{Pyr(Q)}(t) z^t = 1 + \sum_{t\geq 1} (1 + \sum_{j=1}^{t} L_Q(j)) z^t.$$
$$= \sum_{t\geq 0} z^t + \sum_{t\geq 1} \sum_{j=1}^{t} L_Q(j) z^t = \frac{1}{1-z} + \sum_{j\geq 1} L_Q(j) \sum_{t\geq j} z^t.$$
$$= \frac{1}{1-z} + \sum_{j\geq 1} L_Q(j) \frac{z^j}{1-z} = \frac{1 + \sum_{j\geq 1} L_Q(j) z^j}{1-z}.$$

■

The pyramid was constructed over the unit $(d-1)$cube. Thus, using Ehrhart series of unit-cube in terms of Eulerian numbers, Ehrhart series of pyramid is-

$$Ehr_p(z) = \frac{1}{1-z} \sum_{k=1}^{d} \frac{A(d-1,k) z^{k-1}}{(1-z)^d} = \sum_{k=1}^{d} \frac{A(d-1,k) z^{k-1}}{(1-z)^{d+1}}.$$

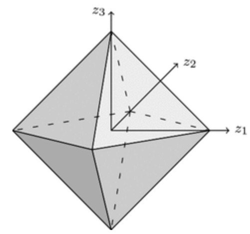

Figure 5. 3-D cross polytope

8. Lattice-point enumeration of Cross-polytopes

Definition 8.1. A cross polytope denoted by $\diamond\, \mathbb{R}^d$ is also known as orthoplex and is symmetric about the origin. Its vertices are the d unit vectors $e_1, e_2, \ldots, e_d$ and their negatives. Its hyperplane description is-

$$:= \{(x_1, x_2, \ldots, x_d) \in R^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}.$$

Figure 5 is a 3 dimensional octahedron.Its vertices are $(\pm1, 0, \ldots, 0), (0, \pm1, \ldots, 0), \ldots, (0, 0, \ldots, \pm1)$. A 2-D cross poltyope is always a square, 3-D a regular octahedron ans 4-D a 16 cell. A cross-polytope's facets are simplexes of previous dimension while its vertex figure is $(d-1)$ dimensional cross-poltyope.

To compute the discrete volume of a cross polytope we use a method similar to that used in section 6. To do this, we define a d dimensional cross-polytope as the bipyramid over a $(d-1)$ dimensional cross polytope Q with vertices $v_1, v_2, \ldots, v_m$ such that Q contains the origin.We define $BiPyr(Q)$, the bipyramid over Q as-

$$conv\{(v_1, 0), (v_2, 0), \ldots, (v_m, 0), (0, \ldots, 0, 1) \textit{ and } (0, \ldots, 0, -1)$$

Thus, the number of lattice points in $tBiPyr(Q)$ is by construction,

$$\begin{aligned} L_{BiPyr(Q)}(t) &= 2 + 2L_Q(1) + 2L_Q(2) + \cdots + 2L_Q(t-1) + L_Q(t). \\ &= 2 + 2\sum_{j=1}^{t-1} L_Q(j) + L_Q(t). \end{aligned}$$

The proof of this identity is similar to the one used for pyramids.

Theorem 8.2. *If Q contains the origin, then $Ehr_{BiPyr(Q)}(z) = \frac{1+z}{1-z} Ehr_Q(z)$. The cross poltyope in dimension 0 is the origin with Ehrhart series $\frac{1}{(1-z)}$. Thus, the higher dimensional cross polytopes can be computed recursively by the formula-*

$$Ehr_\diamond(z) = \tfrac{(1+z)^d}{(1-z)^{d+1}}.$$

Theorem 8.3. *The lattice-point enumerator of a d-dimension cross polytope is,*

$$L_\diamond(t) = \textstyle\sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d} \textit{ for all } t \geq 1.$$

The proof of 8.3 follows from its Ehrhart series.

Proof. Since, $Ehr_\diamond(z) = 1 + \sum_{t\geq 1} L_\diamond(t) z^t$, we can retrieve $L_\diamond$ by expanding $Ehr_\diamond(z)$ into a power series with $z = 0$.

$$\begin{aligned} Ehr_\diamond(z) &= \frac{(1+z)^d}{(1-z)^{d+1}} = \frac{\sum_{k=0}^{d} \binom{d}{k} z^k}{(1-z)^{d+1}} \\ &= \sum_{k=0}^{d} \binom{d}{k} z^k \sum_{t\geq 0} \binom{t+d}{d} z^t. \\ &= \sum_{k=0}^{d} \binom{d}{k} \sum_{t\geq k} \binom{t-k+d}{d} z^t. \end{aligned}$$

Using the fact that $\binom{t-k+d}{d} = 0$ for $0 \leq t < k$ we get,

$$Ehr_\diamond(z) = \textstyle\sum_{t\geq 0} \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d} z^t.$$

Hence,$L_\diamond(t) = \sum_{k=0}^{d} \binom{d}{k} \binom{t-k+d}{d}$. ■

The counting function of cross polytopes bears a connection to the Laugerre polynomials and the Riemann hypothesis.

9. Euler's generating function for Rational Polytopes

We used a counting function for lattice-point enumeration of a standard simplex. We now set up a generating function which can be used for the lattice-point enumerator of any arbitary rational polytope.Such a polytope is given by its hyperplane description, an intersection of half-spaces given by inequalities. We can change this into an equality by introducing a slack variable as we did for the standard simplex. Furthermore, by translation we can assume the polytope has positive coordinates. We can describe every rational polytope P as with some integral matrix $A \in \mathbb{Z}^{md}$ and some integer vector $b \in \mathbb{Z}^m$,

$$P = \{x \in \mathbb{R}^d : Ax = b\}$$

Hence, lattice-point enumerator of tP is the counting function,

$$L_P(t) = \{x \in \mathbb{Z}^d : Ax = tb\}$$

Theorem 9.1. ***Euler's generating function-*** *The lattice-point enumerator of a rational polytope P can be computed by,*

$$L_P(t) = const \left(\tfrac{1}{(1-z^{e_1})(1-z^{e_2})\cdots(1-z^{e_d}) z^{tb}} \right)$$

We can show this by an example.

Example 9.2. *Let P have vertices $(0,0), (2,0), (1,1)$ and $(0, \frac{3}{2})$. Then,*

$$\begin{aligned} L_P(t) &= \left\{ (x_1, x_2) \in \mathbb{Z}^2 : x_1, x_2 \geq 0,\ x_1 + 2x_2, \leq 3t,\ x_1 + x_2, \leq 2t \right\}. \\ &= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : x_1, x_2, x_3, x_4 \geq 0,\ x_1 + 2x_2 + x_3 = 3t,\ x_1 + x_2 + x_4 = 2t \right\}. \end{aligned}$$

The counting function of P is,

$$f(z_1, z_2) := \tfrac{1}{(1-z_1 z_2)(1-z_1^2 z_2)(1-z_1)(1-z_2) z_1^{3t} z_2^{2t}}$$

The geometric series of this equals,

$$f(z_1, z_2) := \left(\sum_{n1 \geq 0} (z_1, z_2)^n)\right) \left(\sum_{n1 \geq 0} ({z_1}^2, z_2)^n)\right) \left(\sum_{n_3 \geq 0} {z_1}^{n_3}\right) \left(\sum_{n_4 \geq 0} {z_1}^{n_4} \frac{1}{{z_1}^{3t} {z_2}^{2t}}\right).$$

$$= \sum_{n_1, \ldots, n_4 \geq 0} {z_1}^{n_1 + 2n_2 + n_3 - 3t} {z_2}^{n_1 + n_2 + n_4 - 2t}$$

The number of lattice points is equal to the number of solutions to $(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4_{\geq 0}$. That is,

Thus,

$$L_P(t) = const \left(\frac{1}{(1 - z_1 z_2)(1 - {z_1}^2 z_2)(1 - z_1)(1 - z_2){z_1}^{3t}{z_2}^{2t}}\right)$$

10. Introducing Cones and Integer-Point Transform

The Ehrhart polynomial of an integral polytope encodes the relationship between its volume and the number of lattice points contained in it. The important question is why the lattice-point enumerator of a polytope is always an Ehrhart polynomial in the first place. The proof of the theorem below provides a precise answer. This fundamental theorem was proved in 1962 by the French mathematician Eugene Ehrhart who made extensive contributions to lattice-point enumeration. Thus, the Ehrhart polynomial and Ehrhart theory are named in his honour.

Theorem 10.1. ***Ehrhart's theorem***- *Given a convex integral polytope $P \in \mathbb{R}^d$, the lattice-point enumerator $L_P(t)$ of P is a rational polynomial in t of degree d which we call the Ehrhart polynomial.*

$$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

Since most proofs work like a charm for a simplex, we first dissect a polytope into simplices.

Definition 10.2. The triangulation of a convex polytope $P \in \mathbb{R}^d$ is the finite collection T of $(d-1)$ simplices such that,

$$P = \bigcup_{\triangle \in T} \triangle.$$

$\triangle_1 \cap \triangle_2$ is a face common of $\triangle_1$ and $\triangle_2$ for every $\triangle_1$, $\triangle_2 \in T$.

Theorem 10.3. *Given a convex polytope $P \in \mathbb{R}^d$ it is always possible to triangulate P using no new vertices.*

This can be proved by lifting P into $\mathbb{R}^{d+1}$. Refer to [DLRS10] for a compete proof of this theorem and further study of triangulations which is an active area of research.

Now we introduce the concept of cones and coning.

Definition 10.4. A cone K in $\mathbb{R}^d$ is a set of points of the form-

$$k = \{v + \lambda_1 w_1 + \lambda_1 w_2 + \cdots + \lambda_m w_m : \lambda_1 + \lambda_2 + \cdots + \lambda_m \geq 0\}.$$

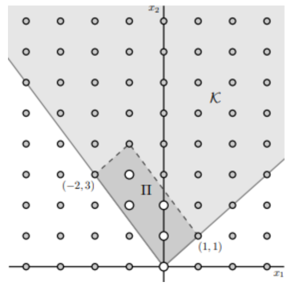

Figure 6. tiling a cone

where $v, w_1, w_2, \ldots, v_n \in \mathbb{R}^d$ and there exists some hyperplane H such that $H \cap K = \{b\}$. v is the vertex of K and w_i are its generators. If K is of dimension d, we call it a d-cone. The d-cone k is simplicial if k has precisely d linearly independent generators. Cone are important for us for the process coning over a polytope.

Definition 10.5. Let $P \in \mathbb{R}^d$ be a convex polytope with vertices $v_1, v_2, \ldots, v_n$. We lift these vertices in $\mathbb{R}^{d+1}$ by adding 1 as their last coordinate. So let,

$$w_1 = (v_1, 1), w_2 = (v_2, 1), \ldots, w_n = (v_n, 1)$$

Now we define the cone over P as,

$$cone(P) = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n : \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \in \mathbb{R}^{d+1}.$$

This pointed cone has the origin as the vertex and we can recover our original polytope P by cutting the cone with hyperplane $x_{d+1} = 1$. Note that like triangulation of a polytope, a cone can be triangulated into simplicial cones using no new generators.

Definition 10.6. Integer-Point Transforms for Rational Cones- Let S be a rational cone or polytope with $S \in \mathbb{R}^d$. Then its integer point transform is a multivariate generating function which encodes the information contained by the lattice points in S. The generating function σ_S is,

$$\begin{aligned} \sigma_S(z) &= \sigma_S(z_1, z_2, \ldots, z_d). \\ &:= \sum_{m \in S \cap \mathbb{Z}^d} z^m, where\ z^m = \Pi_{i=1}^d {z_i}^{d_i}. \end{aligned}$$

Thus, the generating function σ_S simply lists all integer points in S in a special form as a formal sum of Laurent monomials.

Example 10.7. *Consider a 1-dimensional cone $K = [0, \infty)$. Its integer-point transform is,*

$$\sigma_S(z) = \textstyle\sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \frac{1}{(1-z)}.$$

Example 10.8. *To understand obtaining the integer-point transform of a cone K by tiling, we consider the 2-dimensional in figure 6 given by-*

$$k := \{\lambda_1(1,1) + \lambda_2(-2,3) : \lambda_1, \lambda_2 \geq 0\} \in \mathbb{R}^d.$$

We tile K by covering it entirely with translates of the fundamental parallelogram Π-

$$\Pi := \{\lambda_1(1,1) + \lambda_2(-2,3) : 0 \leq \lambda_1, \lambda_2 < 1\} \in \mathbb{R}^d.$$

We first list all vertices of translates of Π using the geometric series with $(j, k \geq 0)$ -

$$\sum_{m=j(1,1)+k(-2,3)} z^m = \frac{1}{(1-z_1z_2)(1-{z_1}^{-2}{z_2}^3)}.$$

Let (m,n) range over $\Pi \cap \mathbb{Z}^2 = \{(0,0),(0,1),(0,2),(-1,2),(-1,3)$ which are the lattice points in the interior of Π. Using (m,n) we generate all lattice points in K apart from its vertices by,

$$L_{(m,n)} := \{(m,n) + j(1,1) + k(-2,3) : j,k \in \mathbb{Z}_{\geq 0}\}$$

So, K is union of $L_{(m,n)}$ and all vertices. Hence,

$$\sigma_K(z) = (1 + z_2 + {z_2}^2 + {z_1}^{-1}{z_2}^2 + {z_1}^{-1}{z_2}^3) \sum_{m=j(1,1)+k(-2,3)} z^m.$$

$$= \frac{(1 + z_2 + {z_2}^2 + {z_1}^{-1}{z_2}^2 + {z_1}^{-1}{z_2}^3)}{(1-z_1z_2)(1-{z_1}^{-2}{z_2}^3)}.$$

Theorem 10.9. *Let K be an integral simplicial d-cone with generators $w_1, w_2, \dots, w_d$. Then for some $v \in \mathbb{R}^d$, the integer point transform of $v + k$ is,*

$$\sigma_{v+k}(z) = \frac{\sigma_{v+\Pi}(z)}{(1-z^{w_1})(1-z^{w_2})\cdots(1-z^{w_d})}.$$

where Π is the fundamental parallelpiped of K with hyperplane description as defined in above example.

Proof. Let $m \in (v+k) \cap \mathbb{Z}^d$ be a lattice point in $(v+k)$. By definition, we have,

$$m = v + \lambda_1 w_1 + +\lambda_2 w_2 + \cdots + +\lambda_d w_d. \tag{10.1}$$

For some unique $\lambda_1, \lambda_1, \dots, \lambda_d \geq 0$. We now decompose each λ_1 into its integer and fractional parts by letting $\lambda_i = [\lambda_i] + \{\lambda_i\}$ for all i with $[\lambda_i]$ as integral and $\{\lambda_i\}$ as fractional.Then (10.1) becomes,

$$m = (v + \{\lambda_1\}w_1 + \{\lambda_2\}w_2 + \cdots + \{\lambda_d\}w_d) + ([\lambda_1]w_1 + [\lambda_2]w_2 + \cdots + [\lambda_d]w_d). \tag{10.2}$$

We can define p as the fractional part by,

$$p = v + \{\lambda_1\}w_1 + \{\lambda_2\}w_2 + \cdots + \{\lambda_d\}w_d.$$

$p \in (v+\Pi) \cap \mathbb{Z}^d$ sine $\{\lambda_1\} < 1$ for all i. We can thus rewrite (10.2) as,

$$m = p + k_1w_1 + k_2w_2 + \cdots + k_dw_d. \tag{10.3}$$

For some unique $k_1, k_2, \dots, k_d \in \mathbb{Z}_{\geq 0}$. Thus, the generating function we have whose coefficients are all $m \in (v+k) \cap \mathbb{Z}^d$ is,

$$\left(\sum_{p\in(v+\Pi)\cap\mathbb{Z}^d} z^p\right)\left(\sum_{k_1\geq 0} z^{k_1w_1}\right)\cdots\left(\sum_{k_d\geq 0} z^{k_dw_d}\right) = \sigma_{v+\Pi}(z)\left(\frac{1}{1-z_1^w}\right)\cdots\left(\frac{1}{1-z_d^w}\right).$$

$$= \frac{\sigma_{v+\Pi}(z)}{(1-z^{w_1})(1-z^{w_2})\cdots(1-z^{w_d})}.$$

■

To complete the puzzle, we need this last theorem before we can prove Ehrhart's theorem.

Theorem 10.10. *For a convex integral polytope $P \in \mathbb{R}^d$, its Ehrhart series can be written as,*

$$Ehr_P(z) = \sigma_{cone(P)}(1, 1, \dots, 1, z_{d+1}).$$

Proof. We first create $cone(P)$ which is the cone over P. As mentioned before, the intersection of $cone(P)$ and the hyperplane $x_{d+1} = 1$ is P itself. So, as a generalization, the intersection of $cone(P)$ and the hyperplane $x_{d+1} = t$ for some positive integer t is tP. Thus, we can decompose P into layers such that each layer is the intersection of P and hyperplane of form $x_{d+1} = t$. Thus, we can rewrite $\sigma_{cone(P)}$ in terms of the layers σ_{tP}'s as,

$$\begin{aligned}\sigma_{cone(P)}(z_1, z_2, \dots, z_{d+1}) &= 1 + \sigma_P(z_1, z_2, \dots, z_d)z_{d+1} + \sigma_{P_2}(z_1, z_2, \dots, z_d)z_{d+1}^2 + \cdots \\ &= 1 + \sum_{t\geq 1} \sigma_{tP}(z_1, z_2, \dots, z_d)z_{d+1}^t.\end{aligned}$$

The 1 terms corresponds to the origin for z_{d+1}^0. Using the fact that, $\sigma_P(1, 1, \dots, 1) = |P \cap \mathbb{Z}^d| = L_P(1)$ which is the lattice-point enumerator of P, this gives,

$$\begin{aligned}\sigma_{cone(P)}(1, 1, \dots, 1, z_{d+1}) &= 1 + \sum_{t\geq 1} \sigma_{tP}(z_1, z_2, \dots, z_d)z_{d+1}^t. \\ &= 1 + \sum_{t\geq 1} L_P(t)z_{d+1}^t. \\ &= Ehr_P(z_{d+1}).\end{aligned}$$

■

11. Main Results of Ehrhart theory

Now we are ready to prove Ehrhart's theorem. Since, every convex polytope $P \in \mathbb{R}^d$ can be triangulated into simplices, we can simply count lattice points in simplices and use inclusion-exclusion to address double-counting. Thus, it suffices to prove Ehrhart theory for simplices.

Lemma 11.1. *If*

$$\textstyle\sum_{t\geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}}.$$

Then f is a polynomial of degree d if and only if-

(1) *$deg\ g \leq d$.*
(2) *$g(1) \neq 0$.*

By this lemma, it suffices to prove that,

$$Ehr_{\triangle}(z) = \frac{g(z)}{(1-z)^{d+1}}. \tag{11.1}$$

for some simplex $\triangle \in \mathbb{R}^d$ where g has degree at most d and $g(1) \neq 0$. Since $\triangle$ is a $d-$simplex, it has exactly $d+1$ vertices which we can denote by $v_1, v_2, \dots, v_{d+1}$. Thus, $cone(\triangle)$ is simplicial as it will have $d+1$ generators denoted by $w_1, w_2, \dots, w_d$. By theorem 10 we have,

$$\sigma_{cone(P)}(z) = \frac{\sigma_\Pi(z)}{(1 - z^{w_1})(1 - z^{w_2}) \cdots (1 - z^{w_{d+1}})}. \tag{11.2}$$

Now if we let $z = (1, 1, \dots, 1, z_{d+1})$ and have $w_i = (\mathrm{v}_i, 1) = (v_{i,1}, v_{i,2}, \dots, v_{i,d}, 1)$ we get that $z^{w_i} = (1^{v_{i,1}})(1^{v_{i,2}}) \cdots (1^{v_{i,d}})(z_{d+1}^1) = z_{d+1}$. Thus, equation 11.1 becomes,

$$\sigma_{cone(\triangle)}(1, 1, \dots, 1, z_{d+1}) = \frac{\sigma_\Pi(1, 1, \dots, 1, z_{d+1})}{(1 - z_{d+1})(1 - z_{d+1}) \cdots (1 - z_{d+1})} = \frac{\sigma_\Pi(1, 1, \dots, 1, z_{d+1})}{(1 - z_{d+1})^{d+1}}. \tag{11.3}$$

Now since equation 11.3 and equation 11.1 have the same form, it remains to prove that $\sigma_\Pi(1,1,\dots,1,z_{d+1})$ is a polynomial of degree at most d and $\sigma_\Pi(1,1,\dots,1,1) \neq 0$. The latter is obvious since,

$\sigma_\Pi(1,1,\dots,1,1) = |\Pi \cap \mathbb{Z}^{d+1}|$. and Π contains the origin so $L_\Pi(1)$ can have min value=1.

Now to prove the former, we recall the definition of integer-point transform which is,

$$\sigma_S(z) = \textstyle\sum_{m \in S \cap \mathbb{Z}^d} z^m.$$

We can now substitute S as Π, $z = (1,1,\dots,z_{d+1})$ and m as $(m_1, m_2, \dots, m_{d+1})$, so we get,

$$\begin{aligned}
\sigma_\Pi(1,1,\dots,1,z_{d+1}) &= \sum_{m \in S \cap \mathbb{Z}^d} (1,1,\dots,1,z_{d+1})^{(m_1,m_2,\dots,m_{d+1})}.\\
&= \sum_{m \in S \cap \mathbb{Z}^d} (1^{m_1})(1^{m_2})\cdots(1^{m_d})(z_{d+1}^{m_{d+1}}).\\
&= \sum_{m \in S \cap \mathbb{Z}^d} (z_{d+1}^{m_{d+1}}).
\end{aligned}$$

Thus, it remains to prove that $m_{d+1} \leq d$ for every $m \in \Pi \cap \mathbb{Z}^d$. Recalling the definition of the fundamental parallelpiped we have,

$$\Pi = \{\lambda_1 w_1 + \lambda_1 w_1 + \cdots + \lambda_{d+1} w_{d+1} : 0 \leq \lambda_i < 1 \; for \; 1 \leq i \leq d+1\}.$$

Since, $w_i = (v_i, 1)$ for all i for all $m = m_1, m_2, \dots, m_{d+1} \in \Pi \cap \mathbb{Z}^d$ we have,

$$m_{d+1} = \lambda_1 + \lambda_2 + \cdots + \lambda_{d+1}.$$

Since, for all i $0 \leq \lambda_i < 1$,

$$m_{d+1} < 1 + 1 + \cdots + 1 = d + 1.$$

Since, m is an integer its max value is d. Thus, the max degree of $\sigma_\Pi(1,1,\dots,1,z_{d+1})$ is d. Thus, finally we have proved Ehrhart's theorem.

Ehrhart Series of an Integeral Polytope- We now build the proof of Ehrhart's theorem further by studying the polynomial $\sigma_\Pi(1,1,\dots,1,z_{d+1})$ to express Ehrhart series in an alternative form.

Corollary 11.2. *Suppose $\triangle$ is an integeral d-simplex with vertices $v_1, v_2, \dots, v_{d+1}$ and let $w_j = (v_j, 1)$. Then,*

$$Ehr_\triangle(z) = 1 + \textstyle\sum_{t \geq 1} L_\triangle(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}}.$$

where h_k^* equals the number of lattice points with last coordinate k. This is because the coefficient of z_{d+1}^k simply counts the integer points in the parallelpiped Π cut with the hyperplane $x_{d+1} = k$. This result can be used to compute $Ehr_\triangle$ of an integral simplex $\triangle$ very quickly.

Stanley's non-negativity theorem shows that $h_0^*, h_1^*, \dots, h_d^*$ are non-negative integers. A full proof can be found in [BD08]. You can also explore an extension of Stanley's theorem in weighted Ehrhart theory in [BDDL$^+$23].

We now present a formula for extracting the Ehrhart polynomial of P from its Ehrhart series as defined above.

Lemma 11.3. *Suppose P is an integral convex $d-$ polytope with Ehrhart series-*

$$Ehr_{\triangle}(z) = 1 + \sum_{t\geq 1} L_{\triangle}(t)z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}}.$$

Then,

$$L_P(t) = \binom{t+d}{d} + h_1^*\binom{t+d-1}{d} + \cdots + h_{d-1}^*\binom{t+1}{d} + h_d^*\binom{t}{d}.$$

The proof of this follows binomial expansion and can be found at [BR07].

Lastly in this section, we shall describe a theorem which beautifully connects the number of integer lattice points in a convex polytope and those which are strictly inside the polytope.

Theorem 11.4. ***Ehrhart Macdonald Reciprocity-*** *Given a convex polytope $P \in \mathbb{R}^d$, evaluating $L_P(t)$ at negative integers yields,*

$$L_P(-t) = (-1)^d L_{P^\circ}(t).$$

Recalling, lattice-point enumeration of the unit d cube and its interior:

$$L_{\square_d}(t) = (t+1)^d.$$
$$L_{\square_d^\circ}(t) = (t-1)^d.$$

Using Ehrhart-Macdonald Reciprocity, we can find $L_{\square_d^\circ}(t)$ directly from $L_{\square_d}(t)$ by evaluating $L_{\square_d}(t)$ at negative integers. This gives,

$$L_{\square_d}(-t) = (-1)^d L_{\square_d^\circ}(t).$$
$$L_{\square_d^\circ}(t) = (-1)^d . L_{\square_d}(-t).$$
$$L_{\square_d^\circ}(t) = (-1)^d . (-t+1)^d.$$
$$L_{\square_d^\circ}(t) = (t-1)^d.$$

A full proof of this theorem can be found at [Mac71] and an alternate proof can be found at [Sam09].

12. Interpreting coefficients of Ehrhart polynomial

It turns out that the coefficients of the Ehrhart polynomial encode some very important information. We aim to decode some of them here.

From Discrete to Continuous volume- We start by presenting a very important theorem which decodes the leading coefficient of an Ehrhart polynomial. The theorem provides a fundamental connection between the discrete and continuous volume of a polytope by enabling us to calculate a polytope's continuous volume from its Ehrhart polynomial.

Theorem 12.1. *For a given convex integral polytope $P \in \mathbb{R}^d$ let its Ehrhart polynomial as defined above be,*

$$L_P(t) = c_d t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + c_0. \textit{ Then } c_d \textit{ equals the volume of } P.$$

Proof. In higher dimensions, the volume of a polytope $P \in \mathbb{R}^d$ denoted by $volP$ can be computed by approximating P with d-dimensional boxes that get smaller and smaller. To be precise, if we take the boxes with side length $1/t$ then each have volume $1/t^d$. We might further think of the boxes as filling out the space between lattice points in the lattice $\frac{1}{t}Z^d$. Thus, if we take the limit as the side length of the d-dimensional approaches 0, we reach the precise value of the continuous volume of P. This is equivalent to counting the number of lattice points inside P on with a smaller and smaller lattice. As such,we can define,

$$volP := \lim_{t\to\infty} \frac{1}{t^d}|P \cap (\tfrac{1}{t}Z)^d|.$$

Since shrinking the lattice by a factor of t is equivalent to expanding P by a factor of t, we can rewrite this definition as,

$$volP := \lim_{t\to\infty} \frac{1}{t^d}|tP \cap Z^d| = \lim_{t\to\infty} \frac{1}{t^d} L_P(t).$$

We now have,

$$\begin{aligned} volP &= \lim_{t\to\infty} \frac{c_d t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + 1}{t^d}. \\ &= \lim_{t\to\infty} (c_d + c_{d-1}t^{-1} + \cdots + c_1 t^{-d+1} + t^{-d}). \\ &= c_d. \end{aligned}$$

■

We will now briefly examine the second leading coefficient of the Ehrhart polynomial.

Theorem 12.2. *Suppose $L_P(t) = c_d t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + c_0$ is the Ehrhart polynomial of an integral polytope P. Then,*

$$c_{d-1} = \frac{1}{2} \sum_{F facet of P} vol(F).$$

This theorem establishes the relationship between the second leading coefficient c_{d-1} of Ehrhart polynomial of P and leading coefficients of the Ehrhart polynomials of the facets of P. The proof of this theorem involves using the Dehn-Somerville relations.

The constant term c_0 of the Ehrhart polynomial is the Euler characteristic of P and is equal to 1. The reader may wonder if the middle coefficients of the Ehrhart polynomial can be decoded as well. These coefficients are much more mysterious and an active area of research.

13. Interpolation

We know use the polynomial behavior of L_P of an integral polytope P to compute the continuous volume $volP$ and the discrete volume L_P from a finite data. A degree-d polynomial is uniquely determined by $d+1$ distinct points. Interpolation involves finding the lattice-point counts for dilates 0 to n which uniquely gives the Ehrhart polynomial of P for all dilates. It is is a useful tool in Ehrhart theory because it can be simpler to just enumerate the first few dilates of a polytope then to find some general formula. In case of an Ehrhart polynomial $L_P(t)$, the interpolation equation is-

$$\begin{pmatrix} L_P(x_1) - 1 \\ L_P(x_1) - 1 \\ \vdots \\ L_P(x_d) - 1 \end{pmatrix} = \begin{pmatrix} x_1^d & x_1^{d-1} & \cdots & x_1 \\ x_2^d & x_2^{d-1} & \cdots & x_2 \\ \vdots & \vdots & & \vdots \\ x_d^d & x_d^{d-1} & \cdots & x_d \end{pmatrix} = \begin{pmatrix} c_d \\ c_{d-1} \\ \vdots \\ c_1 \end{pmatrix}$$

Example 13.1. ***Reeve's tetrahedron-*** *Let T_h be a tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$and $(1,1,h)$ where h is a positive integer.*

To interpolate the Ehrhart polynomial $L_{T_h}(t)$ from its values at various points we deduce the following,

$$4 = L_{T_h}(1) = vol(T_h) + c_2 + c_1 + 1.$$
$$9 = L_{T_h}(2) = vol(T_h).2^3 + c_2.2^2 + c_1.2 + 1.$$

Using the volume formula for a pyramid we know that,

$$vol(T_h) = \frac{1}{3}(base\ area)(height) = \frac{h}{6}.$$

So, $h+1 = h_2c_2 - 1$ which gives $c_2 = 1$ and $c_1 = 2 - \frac{h}{6}$. Thus,

$$L_{T_h} = \frac{h}{6}t^3 + t^2 + (2 - \frac{h}{6})t + 1.$$

14. Extended Ehrhart theory and Quasipolynomials

The central theorem of this paper, theorem 10.1 was based on the Ehrhart polynomial of an integral polytope.In this section, we will give an extension of this theorem to study the lattice-point enumerator of rational polytopes which surprisingly turns out to be quasipolynomial.

Definition 14.1. Quasi-polynomial- The coefficients of a quasi-polynomial are periodic functions with an integral period. Thus, we define a quasipolynomial Q an expression of the form,

$$Q(t) = c_n(t)t^n + \cdots + c_1(t)t + c_0(t).$$

where $c_0, \ldots, c_n$ are periodic functions in t. The degree of Q is n and the least common period of $c_0, \ldots, c_n$ is the period of Q. Alternatively, for a quasipolynomial Q, there exists a positive integer k and polynomials $p_0, p_1, \ldots, p_{k-1}$ such that,

$$Q(t) = \left\{ \begin{array}{ll} p_0(t) & if\ t = 0\ mod\ k \\ p_1(t) & if\ t = 1\ mod\ k \\ \vdots & \\ p_{k-1}(t) & if\ t = k-1\ mod\ k \end{array} \right\}$$

The minimal suck K is the period of Q and for this minimal k, polynomials $p_0, p_1, \ldots, p_{k-1}$ are the constituents of Q.

Example 14.2. *Consider the quasipolynomial* $Q(x) = 5n^3 + [\frac{1}{2}, 2, \frac{1}{3}]_n n^2 + [1, \frac{2}{3}]_n n + [\frac{3}{5}, 4]_n$. *This is a quasi-polynomial of degree 3 and the period is the least common multiple of* $1, 3, 2$ *and 2 which is 6.*

Theorem 14.3. ***Ehrhart's theorem for rational polytopes-*** *If P is a convex rational d-polytope, then $L_P(t)$ is a quasipolynomial in t of degree d. Its period divides the least common multiple of the denominators of the coordinates of the vertices of P.*

We will call the least common multiple of the denominators of the coordinates of the vertices of P the denominator of P. A proof of this theorem in [MS05]. We can also prove this theorem to be an extension of theorem 10.1 as the denominator of an integral polytope P is always 1.

Example 14.4. *We will now find the quasi-polynomial of a polytope P given by,*

$$x \geq 0$$
$$x \leq \frac{n}{3}$$

The vertices of P are 0 and $\frac{n}{3}$, thus the denominator of P is 3. Therefore, the general form of the resulting quasi-polynomial should be,

$$L_P(t) = \alpha t + [\beta_1, \beta_2, \beta_3],$$

To determine α, β_1, β_2 and β_3 one must know some initial values of the quasi-polynomial which are the lattice-point enumerators for some dilates of P. We find that,

t	$L_P(t)$
0	1
1	1
2	1
3	2

From $l_p(0) = 1$ we can calculate $\beta_1 = 1$. Thus, since $L_P(3) = 3\alpha + \beta_1 = 3\alpha + 1 = 2 \to \alpha = \frac{1}{3}$.Solving the remaining two equations gives $\beta_2 = \frac{2}{3}$ and $\beta_3 = \frac{1}{3}$. Thus, the quasi-polynomial in t of degree 1 and period 3 is,

$$L_P(t) = \tfrac{1}{3}t + 1, \tfrac{2}{3}, \tfrac{1}{3}.$$

15. Ehrhart positivity

We now present an open field of research in Ehrhart theory-Ehrhart positivity. A convex integral polytope P is said to have Ehrhart positivity if $L_P(t)$ has all positive coefficients. This gives the central question of this field of research-

Question 15.1. *Which faimilies of integeral polytopes have Ehrhart positivity?*

This turns out to be a challenging question as even though multiple families of polytopes have shown to be Ehrhart positive, the techniques involved are different. Since there is no standard procedure for determining Ehrhart positivity, it continues to be a fascinating area of research in Ehrhart theory.

Products of positive linear polynomials- We present families of polytopes which can be shown to be Ehrhart positive using the following naive lemma.

Lemma 15.2. *Suppose a polynomial $f(t)$ is either*

(1) a product of linear polynomials with positive coefficients, or
(2) a sum of products of linear polynomials with positive coefficients.

Then $f(t)$ has positive coefficients. Using this lemma we can show the two simplest families of polytopes-unit cubes and standard simplex to be Ehrhart positive.

Theorem 15.3. *The unit d-cube is Ehrhart positive.*

Proof. As proved in section 5, $L_{\square}(t) = (t+1)^d$. Since, $(t+1)$ is a linear polynomial with positive coefficients and d is a positive integer, $(t+1)^d$ has positive coefficients as well. ■

Theorem 15.4. *The standard d-simplex is Ehrhart positive.*

Proof. As proved in section 6, $L_{\triangle}(t) = \binom{d+t}{d}$. . Thus, by the expansion of the binomial coefficient we get, ■

$$\binom{d+t}{d} = \frac{(d+t)(d+t-1)\cdots(t+1)}{d!.t!}.$$
$$= \frac{(d+t)!}{d!}.$$

Since $d!$ is a positive constant and the numerator is also a product of linear polynomials with positive coefficients, $\binom{d+t}{d}$ written as a polynomial must have positive coefficients.

There are many other classes of polytopes that are Ehrhart positive including the cross-polytope; however, proving their positivity involves more complex methods. For an in-depth study of Ehrhart positivity, refer to [Liu19]. A unique application of Ehrhart theory is in the Voting theory.

16. Voting Paradox

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