Sieve Methods in Enumerative Combinatorics

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1. Approximating with an **overcount** and then subtracting off an **overcounted approximation of our error**, then overcounting this error, subtracting it off, and so on, until we have a correct count of the objects. This method is what the *Principle of Inclusion-Exclusion* is all about.

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2. The elements of the larger set can be **weighted**, in such a way that unwanted elements cancel out, leaving us with our original set.

Most Well-known Sieve

Suppose we have a list of numbers $(1, 2, \ldots, n)$. We want to find the **number of prime numbers** in this list. We can do this in a **systematic approach** instead of testing out every single number, as you may realize would take quite a while, especially for large n.

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Example: Sieve of Eratosthenes

~~1~~	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	60
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1: Sieve of Eratosthenes (1-100)

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41	42	43	44	45	46	47	48	49	50
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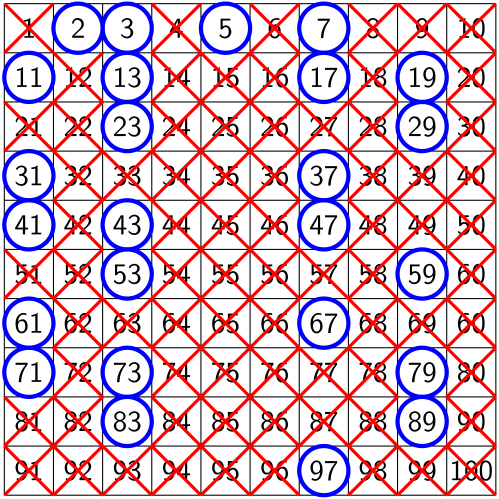

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Overcount?

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For example, let's take primes 2 and 3. 6 is a multiple of both 2 and 3, so we are supposedly "crossing out" 6 once for the round with multiples of 2 and another time with the multiples of 3. This is a clear example of how we can use the Principle of Inclusion-Exclusion to our advantage.

Piece of PIE

Definition

The Principle of Inclusion-Exclusion (PIE) is a combinatorial argument used to find the cardinality of the union of sets by adding and subtracting off the intersections of the subsets.

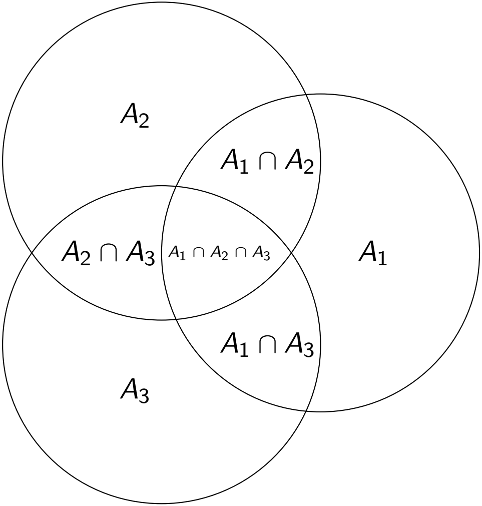
Theorem 2.1

Suppose we have subsets $A_1, A_2, \ldots, A_n$ of set X. Let X be the union of the sets A_i for integers $1 \leq i \leq n$. $|X|$ defined as $|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n|$. Then, $|X|$ can be expressed as

$$|X| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \quad (2.1)$$

Note that X is the alternating sum of the intersections of the subsets.

Simpler Case (with 3 sets)



A_1, A_2, and A_3 are subsets of X. Suppose we are trying to find $|X|$, the **cardinality of the union of the three subsets**. Note that the expression

$$|A_1| \cup |A_2| \cup |A_3|$$

has **overcounted** each $A_1 \cap A_2$, $A_2 \cap A_3$, and $A_1 \cap A_3$ **twice**, and $A_1 \cap A_2 \cap A_3$ **thrice**. Therefore, we must subtract each pairwise intersection and add back the intersection of all three sets.

Proof of Principle of Inclusion-Exclusion

Proof.

Suppose we have an element $j \in X$, that is in k of the m subsets. Suppose we assume that j is in subsets $A_1, \ldots A_k$. This means that j is not in the rest of the subsets $A_{k+1}, \ldots, A_m$.

The following is how many times j is counted:

$$\begin{gathered}\binom{k}{1} \text{ times in } A_1, \ldots, A_k \\ -\binom{k}{2} \text{ times in } A_a \cap A_b \\ +\binom{k}{3} \text{ times in } A_a \cap A_b \cap A_c \\ \vdots \\ +(-1)^{k-1}\binom{k}{k} \text{ times in } A_1 \cap A_2 \cdots \cap A_k \\ = \binom{k}{0} - (1-1)^k = 1.\end{gathered}$$

Therefore, every element $j \in X$ is counted once and only once in the set X for all m, as desired. ∎

Direct Results of Inclusion-Exclusion

Let A be a finite set and let S be a set of properties that may or may not be satisfied by set A. For any $T \subseteq S$, we can denote $f_{=}(T)$ to be the number of objects in A that exactly satisfy the properties in T.
The number of objects in A that at least satisfy the properties in T can be denoted by $f_{\geq}(T)$. Importantly, we get the following relation

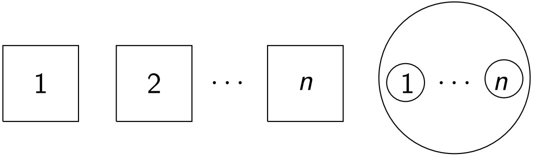
$$f_{=}(\varnothing) = \sum_{U \supseteq T} (-1)^{|U|} f_{\geq}(T). \tag{2.2}$$

This is useful because we can **count the number of objects that satisfy none of the properties** and in turn we can calculate the set of objects that do satisfy the properties. We will use this property in the following example.

Derangements

A very straightforward application of PIE is the *derangement problem*. Let S_n denote the set of permutations of S. Then, how many $\omega \in S_n$ are there such that $\omega(i) \neq i$? This type of permutation is what is known as a derangement. Notice how we can use (2.2).

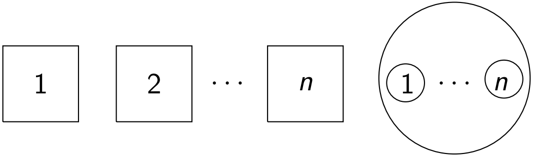
Figure 3: Derangements



Problem Statement

An example of derangements is the number of permutations of n objects into n boxes. Let $D(n)$ denote the number of ways to arrange n objects in n boxes such that no object n_i is put in the ith box.

Figure 4: Derangements



Derangement Formula

Using equation (2.2), we can write

$$D(n) = f_{=}(\varnothing) = \sum_{U \supseteq S} (-1)^{|U|} f_{\geq}(S),$$

where S is the set of properties that $\omega(i) \neq i$ for all $i = 1, 2, \ldots n$. It can be seen from this that

$$D(n) = \sum_{U \supseteq S} (-1)^{|U|} (n - |U|)!.$$

Setting k equal to $|S|$, we get the following equation

$$D(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k!, \qquad (2.3)$$

Derangement Formula

Expanding,

$$D(n) = n!\left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right), \tag{2.4}$$

Taking the limit as n approaches infinity of the previous equation yields an asymptotic formula that may sometimes be preferred over the closed form we have just derived:

$$\lim_{n\to\infty} D(n) = \frac{n!}{e}. \tag{2.5}$$

This equation also gives the closest integer to $D(n)$ for all $n \geq 0$. Furthermore, we can express the closed form as a generating function as follows:

$$\sum_{n\geq 0} D(n)\frac{x^n}{n!} = \frac{1}{e^x(1-x)} \tag{2.6}$$

Restricted Permutations

This is a whole class of problems known as restricted permutations, where there is some condition that causes the number of ways to permute the objects to be "sieved" off.

Ferrers Boards

Boards, Rook Numbers, and Rook Polynomials

Definition 3.1

(Board) Let $[n]$ be $\{1, 2, \ldots, n\}$ Define a Board $B \subseteq [n] \times [n]$.

Definition 3.2

(Rook Number) Define r_k to be the number of k-subsets such that no two elements have a single coordinate in common.

This is corresponding to placing k rooks onto a board B such that no two rooks attack each.
We also define a rook polynomial by

$$r_B(x) = \sum_k r_k x^k.$$

Partitions

An integer *partition* is the number of ways to split a number n into positive integers that add up to n. For example for $n = 4$, we get

$$1 + 1 + 1 + 1$$

$$1 + 1 + 2$$

$$2 + 2$$

$$1 + 3$$

$$4$$

Partitions

Usually, this is simplified to

$$1+1+1+1 = (1,1,1,1) = \boxed{(1^4)}$$

$$1+1+2 = (1,1,2) = \boxed{(1^2,2)}$$

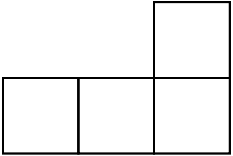
$$2+2 = (2,2) = \boxed{(2^2)}$$

$$1+3 = \boxed{(1,3)}$$

$$4 = \boxed{(4)}$$

Example

Ferrers boards are used to describe the partitions. Each value is depicted by the number of boxes in each row. For example, this is the Ferrers board for $1 + 1 + 2$



The first 2 columns represent the 1's and the last column represents a 2, for a total of 4 squares.

Application

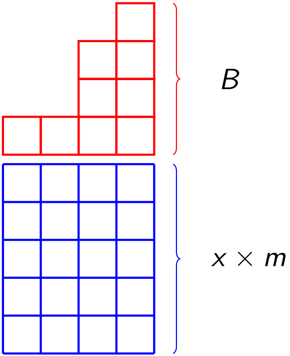
Theorem 3.3

Let $(b_1, b_2, \ldots, b_m)$ *denote the Ferrers board* F*.* $\sum r_k x^k$ *is the rook polynomial for the Ferrers board* F*. If* $s_i = b_i - i + 1$*, then*

$$\sum_k r_k (x)_{m-k} = \prod_{i=1}^{n} (x + s_i) \tag{3.1}$$

Proof of Application

To place k rooks on board B', where B' is the Ferrers board of shape $\{x + b_1, x + b_2, \cdots, x + b_m\}$. Therefore $B' = B \cup C$ where C is the rectangle of dimension $x \times m$ directly underneath B.



Proof of Application

We can count the rook number $r_k(x)_{m-k}$ in two different ways:

1. We can place k rooks on B and then place the remaining $m-k$ rooks in C, such that no rooks attack each other. Placing the rooks on B is equal to r_k and placing the rooks on C is $(x)_{m-k}$, by definition, giving $\boxed{\sum_k r_k(x)_{m-k}}$.
2. There are $x + b_i$ ways to place a rook in the ith column of B, and $x + b_{i-1} - 1$ ways to place a rook in the $(i+1)th$ column, such that it does not attack the first rook, giving us $\boxed{\prod_{i=1}^{n}(x + s_i)}$.

Therefore we have a bijection between our two cases giving us

$$\sum_k r_k(x)_{m-k} = \prod_{i=1}^{n}(x + s_i)$$

Conclusion

Ferrers boards are just one of the many applications of PIE and examples of sieve methods. In my paper, I talk about many of such applications, such as expanding on restricted permutations, how this all ties together with matrices and determinants and its relation with lattice path, as well as involutions, and V-Partitions and Unimodal sequences.

$$\begin{bmatrix} a & b & c & \cdots \\ d & e & f & \cdots \\ g & h & i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$