On results provable with ordinals but unprovable with Peano arithmetic

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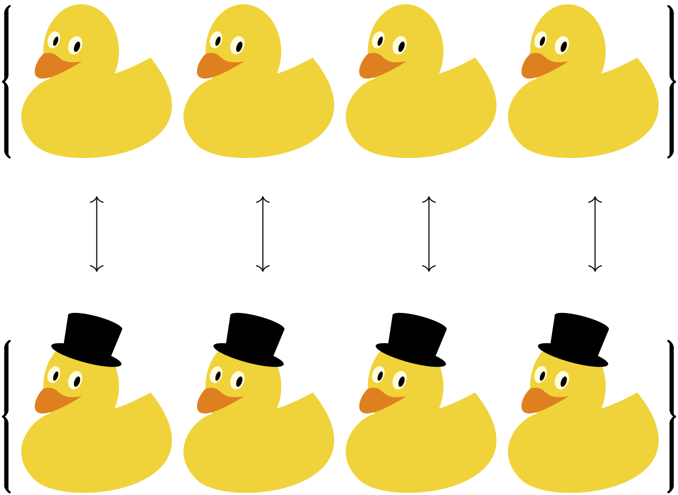
Initial motivations

How do we know how rigid mathematics is?
In the 20th century, David Hilbert's wished that a mathematical system could be shown to be *complete*, *consistent*, and *decidable*. A basic system of mathematics that was based in set theory and first-order logic, known as **Peano arithmetic**, was created to show the rigidity of mathematics. However, beginning with **Gödel's incompleteness theorems**, it was shown that none of the above three were provable.
Ordinal arithmetic, a different mathematical system, was used to prove the following three results, all of which were proven to be unprovable under Peano arithmetic:

- **Gentzen's consistency proof with primitive recursive arithmetic**
- Goodstein's theorem and proof of all Goodstein sequences ending
- Kirby-Paris proof of defeatibility of a graph-theory hydra

Cardinals and cardinality



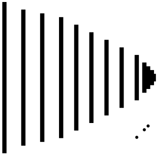
Extending cardinality to infinity

$\mathbb{N}$	0	1	2	3	4	$\cdots$	n_k
$\mathbb{Z}$	0	-1	1	-2	2	$\cdots$	$(-1)^{n_k} \cdot \lceil n_k/2 \rceil$

Table 1: The set of natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$ have the same cardinality $\aleph_0$.

Figure 1: The cardinality of these sticks is $\aleph_0$.

How do we count this?



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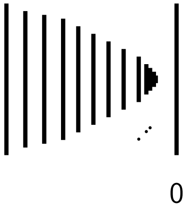

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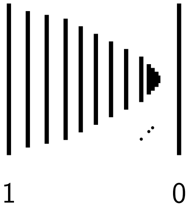

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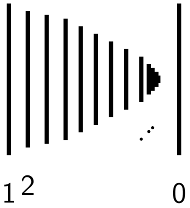
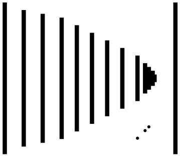


How do we count this?

Figure 2: The cardinality of these sticks is *still* $\aleph_0$.

How do we *order* this?



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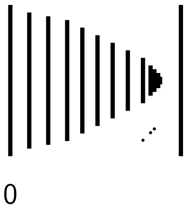

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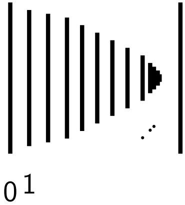

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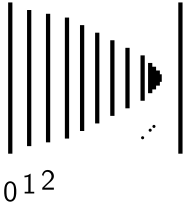

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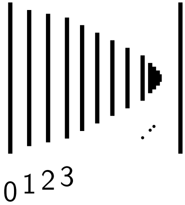

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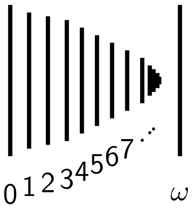

Figure 3: The cardinality of these sticks is *still* $\aleph_0$, even though we have ordered the set in a different way.

Several helpful definitions

Definition (Well-ordered set)

A set S is defined to be *well-ordered* if it satisfies both of the following two properties:

1. S is totally ordered (all elements in S are comparable with each other; this is often denoted as $(S, \leq)$).
2. Every nonempty subset of S has a least element. The set of integers $\mathbb{Z}$ is not *well-ordered* because it has no least element.

Definition (Order type)

Two *well-ordered* sets A and B have the same *order type* if there exists a bijection $b : A \to B$ that orders the A and B with the same *ordinal*.

Extension to more ω's

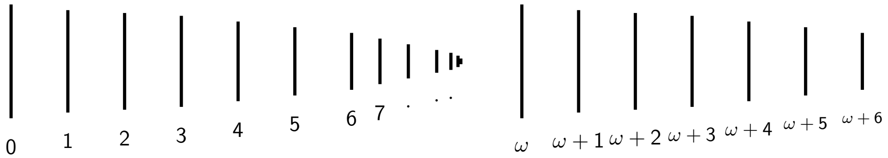

Figure 4: The *cardinality* of these sticks is *still* $\aleph_0$, even though we have ordered the set in a different way. The *order type* of this set is $\omega + 7$.

Transfinite induction

The ability to determine properties of higher ordinals based on the ones "counted" so far is an important part of ordinal proofs. Gentzen's proof very clearly uses this.

Transfinite induction involves the following process:

1. Some property $\mathcal{P}(\alpha)$ is defined for all ordinals α.
2. Whenever $\mathcal{P}(\beta)$ is true for all $\beta < \alpha$, then $\mathcal{P}(\alpha)$ is also true.
3. For a limit ordinal λ, $\mathcal{P}(\lambda)$ is true given $\mathcal{P}(\beta)$ is true for all $\beta < \lambda$.

Premise of **Gentzen's consistency proof**

In 1900, Hilbert put forward his second problem, asking for a proof that arithmetic is consistent (free of internal contradictions). For formal arithmetic system $\mathcal{F}$, we can create some formula $Cons(\mathcal{F})$ that describes the consistency of $\mathcal{F}$.

In 1931, Kurt Gödel published his second incompleteness theorem that proved that for any system $\mathcal{F}$, $Cons(\mathcal{F})$ cannot be proved under it.

Gerhard Gentzen published a paper in 1936 showing how Peano arithmetic does not contain contradictions (i.e. it is consistent), provided that another system, **primitive recursive arithmetic** (PRA for short) bundled with acceptance of transfinite induction up to ε_0 (the limit of the expression $\omega^{\omega^{\omega^{\cdots}}}$), does not contain contradictions (i.e. it is also consistent).

Overview of Gentzen's steps

1. Any value of truth for the consistency of any "derivation" is "reduced" to the values of truth for the consistencies of simpler derivations that led to the previous derivation.
2. A transfinite ordinal number is associated with every derivation and for each reduction. A contradiction (proving inconsistency) is assigned a corresponding *smaller* ordinal number than the ones above it. This is shown to be impossible in PRA.
3. Therefore, Peano arithmetic cannot contain contradictions, QED.

Gentzen's consistency proof, Step 1 - Notation

Gentzen's proof relies on a system known as *LK*-calculus, which is reliant on notation of certain logical implications known as *sequents*. The most important notational notes are listed.

1. Predicate symbols used to define logic, such as $\supset$, are not used, replaced with the logic symbols $\wedge$, $\neg$, $\vee$.
2. A *formula* is a set of arbitrary terms (like 1, $1''$, and α) connected with *predicate symbols* (like $\wedge$, $\neg$, $\vee$, $\exists$, and $\forall$).
3. A *sequent* is an expression of the form $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m \implies \mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$ for some natural numbers m, n. All the $\mathcal{A}$'s are known as the *antecedent formulae* and all the $\mathcal{B}$'s are known as the *succedent formulae*. Any antecedent and any succedent can be empty, and each formula's truth value is known.
4. Any sequent is false if all antecedent formulae are true and all succedent formulae are false. $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m \implies \mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n$ can be read as "If the assumptions $\mathcal{A}$ are true, then at least one of the propositions $\mathcal{B}$ holds.
5. An *inference figure* involves an *upper sequent*, a *lower sequent*, and a *line of inference* written between them.
6. The *structural inference figures* demonstrate four different rules, shown in Figure 5.
7. The *operational inference figures* (not shown) code the five different logical connectives ($\wedge$, $\vee$, $\forall$, $\exists$, $\neg$) with mathematical implication relationships.
8. *CJ-inference figures* are what Gentzen calls "the formal counterparts of complete inductions". These follow the format $\frac{\mathcal{F}(a),\mathbb{P} \implies \mathbb{Q},\mathcal{F}(a)'}{\mathcal{F}(1),\mathbb{P} \implies \mathbb{Q},\mathcal{F}(t)}$, where a is a free variable and t ranges from 1 up to a.

$$\textbf{Thinning}: \frac{\mathbb{P} \implies \mathbb{Q}}{\mathcal{A},\mathbb{P} \implies \mathbb{Q}} \text{ and } \frac{\mathbb{P} \implies \mathbb{Q}}{\mathbb{P} \implies \mathbb{Q},\mathcal{A}}$$

$$\textbf{Contraction}: \frac{\mathcal{A},\mathcal{A},\mathbb{P} \implies \mathbb{Q}}{\mathcal{A},\mathbb{P} \implies \mathbb{Q}} \text{ and } \frac{\mathbb{P} \implies \mathbb{Q},\mathcal{A},\mathcal{A}}{\mathbb{P} \implies \mathbb{Q},\mathcal{A}}$$

$$\textbf{Interchange}: \frac{\mathbb{O},\mathcal{A},\mathcal{B},\mathbb{P}, \implies \mathbb{Q}}{\mathbb{O},\mathcal{B},\mathcal{A},\mathbb{P} \implies \mathbb{Q}} \text{ and } \frac{\mathbb{P} \implies \mathbb{Q},\mathcal{A},\mathcal{B},\mathbb{R}}{\mathbb{P} \implies \mathbb{Q},\mathcal{B},\mathcal{A},\mathbb{R}}$$

$$\textbf{Cut}: \frac{\mathbb{P} \implies \mathbb{Q},\mathcal{A} \quad \mathcal{A},\mathbb{O} \implies \mathbb{R}}{\mathbb{P},\mathbb{O} \implies \mathbb{Q},\mathbb{R}}$$

Figure 5: Four different things you can do with structural inference structures.

Gentzen's consistency proof, Step 1 - Formulation

Lacking antecedent formulae before $\implies$ signifies that the succedent formulae are always true no matter the assumptions, and lacking succedent formulae after $\implies$ signifies that the assumptions made in the antecedent formulae create a contradiction.

Thus, an empty sequent, one with no antecedent or succedent formulae, demonstrates that without any prior assumptions, a contradiction will result in a system.

In other words, our proof of the consistency of Peano arithmetic must involve proving that an empty sequent *cannot* be found or deduced through a derivation.

Gentzen's consistency proof, Step 2 - Ordinal association

Condition	**Line of inference's order**
Structural inference figure	$\eta(S_{upper})$
Structural inference figure with cut	$\eta(S_{upper_1}) \# \eta(S_{upper_2})$
Operational inference figure	$\eta(S_{upper}) + 1$
Operational inference figure with two sequents	$max(\eta(S_{upper_1}), \eta(S_{upper_2})) + 1$
CJ-inference figure	$\omega^{\alpha_{max}+1}$

Condition	**Lower sequent's order**
$\phi(S_{lower}) = \phi(S_{upper})$	$\eta(L)$
$\phi(S_{lower}) = \phi(S_{upper}) - 1$	$\omega^{\eta(L)}$
$\phi(S_{lower}) = \phi(S_{upper}) - 2$	$\omega^{\omega^{\eta(L)}}$
$\phi(S_{lower}) = \phi(S_{upper}) - 3$	$\omega^{\omega^{\omega^{\eta(L)}}}$
$\phi(S_{lower}) = \phi(S_{upper}) - n$	$\omega^{\omega^{\cdot^{\cdot^{\cdot \eta(L)}}}}$, for n ω's

Table 2: Let $\eta(S)$ be the ordinal corresponding to a certain sequent S, and let $\eta(L)$ be the ordinal that corresponds to a certain line of inference L. α_{max} represents the largest ordinal power of η, and $\phi(S)$ represents the level (number of logical connectives and quantifiers) of a certain sequent S.

Gentzen's consistency proof, Step 3 - Contradiction if there is a contradiction

Gentzen goes on to create a specific reduction formula (a special kind of cut-elimination) for each of the five logical connectives and shows how each one can be used with a CJ-inference figure to decrease the size of the ordinal.

Using this, we achieve an infinitely and strictly descending sequence of ordinals all less than ε_0 corresponding to a formula that can recursively act on every single possible sequent in elementary or Peano arithmetic.

If we presume an elementary arithmetic system such as Peano arithmetic has a contradiction, we can prove in the system of PRA that there would be a contradiction in PRA and LK-calculus.

Result of this proof

Main takeaways:

- Ordinal arithmetic, coupled with primitive recursive arithmetic and transfinite induction up to ε_0, proved that Peano arithmetic is consistent.
- This proof is controversial due to the usage of infinite ordinals up to a limit, ε_0.
- For any two formal systems $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B}$ can be considered *stronger* than $\mathcal{A}$ if $\mathcal{B}$ can prove a wider range of mathematical statements compared to $\mathcal{A}$. The PRA and ε_0 system that Gentzen used turns out to be incomparable with Peano arithmetic.
- At the end of the day, mathematicians are still arguing over the interpretation of these proofs.
- Ordinal arithmetic, due to the powerful property of transfinite induction, seems to be quite formidable and powerful as a tool of proof.