Symmetric Polynomials

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Some Definitions

Field

A field is a commutative ring with unity (meaning that it has an identity element) where every element that is not zero is a unit.

Unit

It means having the multiplicative inverse,

$$\forall a \in \mathbb{R}, \exists a^{-1} \in \mathbb{R} \tag{1}$$

Unity

It means having an additive and multiplicative identity.

Polynomial Ring

The Polynomial Ring in N Variables

Suppose we have $x_1, \ldots, x_n$ that are called variables. So we can think of a polynomial from $x_1, \ldots, x_n$.
We also define $F[x_1, \ldots, x_n]$ as a set of all polynomials in $x_1 \ldots x_n$ with coefficients from F.

Theorem

Theorem

If we have field F a ring R that contains F, an $a_1 \dots a_n \in R$ then

$$f(x_1 \dots x_n) \mapsto f(a_1 \dots a_n) \tag{2}$$

is a ring homomorphism.

Example

Consider following polynomial $f(x)x^4 + r_1x^3 + r_2x^2 + r_3x + r_4 \in F$ has roots $r_1, r_2, r_3, r_4 \in F[x]$ as well. Then,

$$f = (x - r_1)(x - r_2)(x - r_3)(x - r_4) \tag{3}$$

If we multiply this out the coefficients can be expressed as,

$$r_1 = -(r_1 + r_2 + r_3 + r_4), \tag{4}$$
$$r_2 = (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4), \tag{5}$$
$$r_3 = -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4), \tag{6}$$
$$r_4 = r_1r_2r_3r_4. \tag{7}$$

Elementarty Symmetric Polynomials

Consider variables $x_1 \dots x_n$ that are distinct variables then,

$$\sigma_1 = x_1 + x_2 \cdots + x_n, \tag{8}$$

$$\sigma_2 = x_1x_2 + x_1x_3 \dots x_{n-1}x_n, \tag{9}$$

$$\vdots \tag{10}$$

$$\sigma_r = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1}x_{i_2} \dots x_{i_r}, \tag{11}$$

$$\sigma_n = x_1 \dots x_n \tag{12}$$

are called **elementary symmetric polynomials.**

Key Identity of Elementary Symmetric Polynomials

Proposition

Suppose we have variables $x_1, \ldots, x_n$ over field F and another variable x we have,

$$(x-x_1)\ldots(x-x_n) = x^n - \sigma_1 x^{n-1} + \cdots + (-1)^r \sigma_r x^{n-r} + \cdots + (-1)^n \sigma_n \quad (13)$$

Proof

Proof.

Start by multiplying out the left side of the equation (13) we will get the coefficient for each power of x. It follows that the term x_{n-r} are the product of where we chose x exactly $n - r$ times. That means that choosing $-x_1$ for the $i_1 st, \dots i_r th$ factors and choosing x for remaining n-r factor(s). The product of these choices is

$$(-x_{i_1})(-x_{i_2})\dots(-x_{i_r})x^{n-r} = (-1)^r x_{i_1}x_{i_2}\dots x_{i_r}x^{n-r}. \tag{14}$$

Sum over all possible ways of making the n choices we get that coefficient of x^{n-r} on the left-hand side as,

$$(-1)^r \sum_{1\leq i_1<\dots<i-r\leq} x_{i_1}\dots x_{i_r} = (-1)^r \sigma_r. \tag{15}$$

□

Application of the Proposition

Suppose f is a monic polynomial,

$$f = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in F[x]. \tag{16}$$

That it has roots $a_1, \ldots, a_n$. This means that,

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = (x - a_1)\ldots(x - a_n). \tag{17}$$

Application of the Proposition

Then we can evaluate,

$$(x - a_1)\dots(x - a_n) = x^n - \sigma_1(a_1, \dots, a_n)x^{n-1} + \dots + (-1)^{n-1}\sigma_{n-1} \tag{18}$$

$$(a_1 \dots a_n)x + (-1)^n\sigma_n(a_1, \dots, a_n) \tag{19}$$

The coefficient of f can be written in terms of the roots $a_r = (-1)^r\sigma_r(a_1, \dots, a - n)$.

Symmetric Polynomials

Definition

A polynomial $f \in F[x_1, \ldots, x_n]$ is a **symmetric polynomial** if,

$$f(x_{\alpha(1)}, \ldots x_{\alpha(r)}) = f(x_1, \ldots, x_n) \tag{20}$$

for all permutations in S_n.

Fundamental Theorem of Symmetric Polynomials

Theorem

Any symmetric polynomial in $F[x_1, \ldots, x_n]$ can be written as a polynomial in $\sigma_1, \ldots, \sigma_n$ with coefficients from F.

Applications

- Polynomial factorization
- Combinatorics
- Representation theory
- Geometry
- Number Theory
- Symmetry analysis