Symmetric and Schur Polynomials

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Abstract

This paper serves as a helpful guide on exploring symmetric polynomials and Schur functions.

1 Some Important Definitions

Definition 1.1. A Ring is a nonempty set R that provides two binary operations $\oplus$ and $\otimes$ that have to satisfy all the conditions following.

For all $a, b, c \in R$

1. $a \oplus b \in R$
2. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
3. $a \oplus b = b \oplus a$
4. Element $0_R \in R$ such that,
$$a \oplus 0_R = a, \text{ for all } a \in R. \tag{1}$$
5. For all $a \in R$, the equation
$$a \oplus x = 0_R \tag{2}$$
6. If $a \in R$, and $b \in R$ then $a \otimes b \in R$.
7. $a \otimes (b \otimes c) = (a \otimes c) \otimes b$
8. $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

Definition 1.2. Lexicographic order especially in our case(, $x_1^{a_1} \dots x_1^{a_n} < x_1^{b_1} \dots x_n^{b_n}$ is equal to
$$\exists j \in [1, n], (\forall i \in N, 1 \leq i < j \Rightarrow a_i = b_i), a_j < b_j. \tag{3}$$

2 The Polynomial Ring in N Variables

Let x $\in x_1, \ldots, x_n$, be distinct variables. Suppose we have a polynomial in $x_1, \ldots, x_n$ with coefficients in a field F is a finite sum of terms that are expressions of the form

$$\mathrm{c}x_1^{a_1} \ldots x_n^{a_n}, \mathrm{c \text{ in } F}, a_1 \ldots a_n \geq 0 \text{ in Z}$$

Multiplication of $x_1^{a_1} \ldots x_n^{a_n}$ is monomial so that a term is an element of F times a monomial. A term is nonzero if the constant is nonzero.

Definition 2.1. A nonconstant polynomial F$[x_1 \ldots x_n]$ is irreducible over F if it is not the product of nontrivial polynomials with a smaller degree than the polynomial.

Definition 2.2. An integral domain R is a **Unique Factorization Domain** if,

1. All nonzero element of R is either a unit or a product of irreducibles.
2. If $r_1 \ldots r_k = s_1 \ldots s_l$ where $r_1 \ldots r_k$ and $s_1 \ldots s_l \in$ R are irreducible then k=l, and there is a permutation such that for each 1≤ i≤ k there is a unit $a_i \in$ R such that $r_\cdot i = a_i S_{\sigma(i)}$.

Corollary 2.3. If F is Field then F$[x_1 \ldots x_n]$ is a Unique Factorization Domain.

Theorem 2.4. Let f $\in$ F$[x_1, \ldots x_n]$ be nonconstant. Then there are irreducible polynomials $\alpha_1 \ldots \alpha_n \in$ F$[x_1, \ldots x_n]$ such that

$$f = g_1 \ldots g_n \tag{4}$$

If there is a second factorization of f into irreducibles

$$f = h_1 \ldots h_j \tag{5}$$

then $r = j$ and the h_i's can be permuted so that each h_i is a constant multiple of g_i.

Proof. In the terminology of Unique Factorization Domain in Definition 1.2, this theorem states that F$[x_1 \ldots x_n]$ is a Unique Factorization Domain, see Corollary 2.0.1 □

Theorem 2.5. Given a field F, a ring R containing F, and $a_1 \ldots, a_n \in R$, the evaluation map (2.2) is a ring homomorphism F$[x_1 \ldots x_n] \rightarrow R$.

Proof. We can prove it by verifying,

$$(f+g)(a_1 \ldots, a_n) = f(a_1 \ldots, a_n) + g(a_1 \ldots, a_n), \tag{6}$$

$$(fg)(a_1 \ldots, a_n) = f(a_1 \ldots, a_n) g(a_1 \ldots, a_n), \tag{7}$$

Also clearly $f+g$ denotes sum of polynomials and fg denotes the product of polynomials. □

3 The Elementary Symmetric Polynomials

Definition 3.1. Let $x_1 \ldots, x_n$ be variables over a field F then

$$\sigma_1 = x_1 + \cdots + x_n \tag{8}$$

$$\sigma_2 = \sum_{1 \leq i < j \leq n} x_i x_j, \tag{9}$$

$$\vdots \tag{10}$$

$$\sigma_r = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \ldots x_{i_r}, \tag{11}$$

$$\vdots \tag{12}$$

$$\sigma_n = x_1 \ldots x_n \tag{13}$$

are the **Elementary Symmetric Polynomials**.

Proposition 3.2. *Suppose $x_1 \ldots x_n$ be variables over field F. Then in field F, for variable x we will have*

$$(x - 1) \ldots (x - x_n) = x^n - \sigma_1 x^{n-1} \ldots (-1)^n - \sigma_n \tag{14}$$

Proof. Given the equation in (11), we can see that the constant term is actually just $(-x_1) \ldots (-x_n) = (-1^n)\sigma_n$. Similarly, the coefficient of x^{n-1} is easily seen to be $-x_1 - \cdots - x_n = -\sigma_1$. □

- *For each of n factors $x - x_1$, choose x or $-x_1$.*
- *Take the product of n of them.*
- *Sum these products over all possible ways of making n choices*

Suppose we have the term x^{n-p} in the (11) ones that have x^{n-p} we would need to take the product of them $n-p$ times. With the second bullet, this means that product of those choices is

$$(-x_{i_1}) \ldots (-x_{i_p}) x^{n-p} = (-1)^p x_{i_1} \ldots x_{i_p} x^{n-p} \tag{15}$$

When we sum up all possible choices with n that we can make as emphasized in the third bullet, the coefficients of term $x^n - p$ in the left hand of (11) becomes

$$(-1)^r \sum_{1 \leq i_r < \cdots < i_p \leq n} x_{i_1} \ldots x_{i_p} = (-1)^r \sigma_r. \tag{16}$$

and this completes the proof.

4 Symmetric Polynomials

Definition 4.1.

Symmetric Polynomials are exactly what the name says. If we change any of the variables of a function, we will get the same polynomial.

Definition 4.2. A polynomial $f \in F[x_1, \ldots, x_n]$ is a **Symmetric Polynomial** if

$$f[x_{\tau(1)}, \ldots, x_{\tau(n)}] = f[x_1, \ldots, x_n] \tag{17}$$

for all permutations τ.

Example 4.3. *Let's say we have a polynomial f such that, for permutations $r = (345)$ in the following polynomial*

$$f = x_1^2 + x_2^2 + x_3^2 \tag{18}$$

It can be seen that no matter how many times we interchange the variables we will get the same polynomial.

5 The Fundamental Theorem of Symmetric Polynomials

In Chapter 3 we showed that $\sigma_1 \ldots \sigma_n$ is symmetric, which gave us some clue about what elementary symmetric polynomials are, now we will focus on how they are related to symmetric polynomials other than the property of being symmetric. To be given a brief history fundamental theorem of symmetric polynomials has its roots in the works of Evariste Galois, Augustin-Louis Cauchy, and Joseph-Louis Lagrange. Before starting the proof, I want to mention that proof below belongs to mathematician David Cox in his book of Galois theory.

Theorem 5.1. Any symmetric polynomial in polynomial set $F[x_1 \ldots x_n]$ can be written in terms of elementary symmetric polynomials.

The proof involves an inductive process it requires us to order monomials $x_1^{a_1} \ldots x_n^{a_n}$ in $x_1, \ldots x_n$. Using graded lexicographic order which is,

$$x_1^{a_1} \ldots x_n^{a_n} < x_1^{b_1} \ldots x_n^{b_n} \iff a_1 + \ldots a_n < b_1 + \cdots + b_n, \tag{19}$$

$$\text{or } a_1 + \cdots + a_n = b_1 + \cdots + b_n, a_1 < b_1, \tag{20}$$

$$\text{or just the first terms are equal, } a_1 + \cdots + a_n = b_1 + \cdots + b_n, \text{where,} \tag{21}$$

$$a_1 = b_1 \text{ also } a_2 < b_2 \text{ and so on.} \tag{22}$$

If we were to compare monomials, we would first compute total degree of each monomials and in case these are equal we check two monomials and their exponent at a time. Examples can be given as follow.

$$x_1^5 x_2 x_3 < x_1^2 x_2^3 x_3^3, \text{ because total degree is just smaller.} \tag{23}$$

$$x_1^4 x_2 x_3^2 < x_1^4 x_2^3 x_3, \text{ because } x_2 \text{ degree is just smaller.} \tag{24}$$

Proof. We can now start by let $f \in F[x_1 \dots x_n]$ be a symmetric polynomial with coefficient c

$$cx_1^{a_1} \dots x_n^{a_n}. \tag{25}$$

We can say that

$$a_1 \geq a_2 \dots \geq a_n. \tag{26}$$

We can show it by remarking that f is a symmetric polynomial, which means that even if we change any two or more variables we will get the same polynomial thus it can be written in way like,

$$cx_1^{a_1} \dots x_{i+1}^{a_i} x_i^{a_{i+1}} \dots x_n^{a_n} \tag{27}$$

Again with regard to the definition in Chapter (1) **Lexicographic** order is a total order on the set of monomials, it helps us on determining the leading term of polynomials. It is important to note that both monomials have the same total degree, considering (21,19). Then this implies that $a_{i+1} > a_i$, which again means that it is greater than the relationship in (not added soon, order lexicographic). But there is a contradiction because the leading term is (19) so proved by contradiction. Now we can consider the following elementary symmetric polynomial l,

$$l = \sigma_1^{a_1-a_2} \sigma_2^{a_2-a_3} \dots \sigma_n^{a_n}. \tag{28}$$

This is a polynomial that is in the form of (20). Notice that leading term of σ_r is $x_1 \dots x_r$, it follows that leading term of l is,

$$x_1^{a_1-a_2}(x_1x_2)^{a_2-a_3}(x_1x_2x_3)^{a_3-a_4} \dots (x_1 \dots x_n)^{a_n} \tag{29}$$

$$= x_1^{a_1-a_2+a_2-a_3+\dots+a_n} x_2^{a_2-a_3+\dots+a_n} \dots x_{n-1}^{a_{n-1}-a_n+a_n} x_n^{a_n} \tag{30}$$

$$= x_1^{a_1} \dots x_n^{a_n}. \tag{31}$$

Don't forget to do the distribution in Chapter (3) we have given examples of elementary symmetric polynomials in there. With (31) we actually showed that f and cl have same leading term, according to ordering in (19). Hence $f_1 = f - cl$ has smaller leading term with coefficient c_1 and exponents $b_1 \geq \dots \geq b_n$.

With repeating the same evaluation f_1 instead of f. We can conclude f_1 is symmetric as well and now has coefficient c_1 with exponents $b_1 \geq \dots \geq b_n$. It is like a recursive calculation, which gives us an expression g_1 in the elementary symmetric polynomials such that f_1 and c_1g_1 have same leading terms.

Which follows that,

$$f_2 = f_1 - c_1l_1 = f - cl - c_1l_1. \tag{32}$$

As we did before with continuing the process we get,

$$f_1 = c - l, f_2 = f - cl - c_1l_1, f_3 = f - cl - c_1l_1 - c_2l_2, \dots f_n = f - cl = c_1l_2 \dots. \tag{33}$$

In each stage we get smaller leading term. This process ends if we find some j that $f_j = 0$. On the other hand since we never have such f, that would give us an infinite sequence of nonzero polynomials with strictly decreasing leading terms. The thing is above it has shown that there are only finitely many monomials strictly smaller than the leading term of f. Hence above process must stop at some point.

For that some j, we obtain

$$f = cl + c_1 l_1 + \dots. \tag{34}$$

since $f_m = f - cl - c_1 l_1 - \dots - c_{m-1} l_{m-1}$. Each cl_i is product of the σ_j to various powers, which proves that f is an elementary symmetric polynomial. And this was what we wanted to show which completes the proof.

□

Example 5.2. *Consider the polynomial in x_1, x_2, x_3, x_4, can be given,*

$$f = \sum_4 x_1^3 x_2^2 x_3, \tag{35}$$

this has 24 terms, David cox used that notation for what we did in Chapter(3), meaning that using permutation. Edward Waring, one of the first mathematicians that studied symmetric polynomials. In his book Meditationes Algebraice [6], *he did really similar to what David Cox did in his proof of FTSP. We can proceed as follows,*

Step 1. Considering the leading term of f, $x_1^3 x_2^2 x_3 = x_1^3 x_2^2 x_3^1 x_4^0$, so that (28) becomes,

$$\sigma_1^{3-2} \sigma_2^{2-1} \sigma_{1-0} \sigma_4^0 = \sigma_1 \sigma_2 \sigma_3. \tag{36}$$

So we can show that,

$$\sigma_1 \sigma_2 \sigma_3 = \sum_4 x_1^3 x_2^2 x_3^4 + 3 \sum_4 x_1^3 x_2 x_3 x_4 + 3 \sum_4 x_1^2 x_2^2 x_3^2 + 8 \sum_4 x_1^2 x_2^2 x_3 x_4. \tag{37}$$

$$f_1 = f - \sigma_1 - \sigma_2 - \sigma_3 = -3 \sum_4 x_1^3 x_2 x_3 x_4 - 3 \sum_4 x_1^2 x_2^2 x_3^2 - 8 \sum_4 x_1^2 x_2^2 x_3 x_4. \tag{38}$$

Step 2. The leading term of f_1 is $-3x_1^3 x_2 x_3 x_4$, and from that we get,

$$\sigma_1^{3-1} \sigma_2^{1-1} \sigma_3^{1-1} \sigma_4^1 = \sigma_1^2 \sigma_4 = \sum_4 x_1^3 x_2 x_3 x_4 + 2 \sum_4 x_1^2 x_2^2 x_3 x_4. \tag{39}$$

$$f_2 = f - \sigma_1 \sigma_2 \sigma_3 + 3\sigma_1^2 \sigma_4 = -3 \sum_4 x_1^2 x_2^2 x_3 x_4. \tag{40}$$

Step 3. Again when we expand (40) we get leading term of f_2 as $-3x_1^2 x_2^2 x_3^2$. We can obtain

$$f_3 = f - \sigma_1 \sigma_2 \sigma_3 + 3\sigma_1^2 \sigma_4 = 4 \sum_4 x_1^2 x_2^2 x_3 x_4. \tag{41}$$

From that we see that leading term of f_3 is $4x_1^2 x_2^2 x_3 x_4$,

$$\sigma_2 \sigma_4 = \sum_4 x_1^2 x_2^2 x_3 x_4, \tag{42}$$

we see that.

$$f_4 = f - \sigma_1 \sigma_2 \sigma_3 + 3\sigma_1^2 \sigma_4 + 3\sigma_3^2 - 4\sigma_2 \sigma_4 = 0. \tag{43}$$

Conclusion. Since $f_4 = 0$, the process terminates and we obtain the formula,

$$f = \sigma_1 \sigma_2 \sigma_3 + 3\sigma_1^2 \sigma_4 + 3\sigma_2 \sigma_4. \tag{44}$$

That means that we expressed f in terms of elementary symmetric polynomial σ_n.

6 The Roots of a Polynomial

In Galois theory, symmetric polynomials are often evaluated at the roots $r_1, \dots, r_n$ of polynomial f.

Corollary 6.1. Letting f be a polynomial in polynomial ring, and the polynomial is monic with degree more more than zero, with roots $r_1, \dots, r_n$, in the field L. Uhhh that was long ! For given symmetric polynomial $s(x_1, \dots, x_n)$, that is in field F. Then we will have,

$$s(r_1, \dots, r_n) \in F. \tag{45}$$

Proof. Again recalling the evaluation map $F[x_1, \dots, x_n] \to L$ is defined by p$\to p(r_1, \dots, r_n)$. This is actually a ring homomorphism. Which is proved in theorem (2.5).

Since s is symmetric polynomial in $x_1, \dots, x_n$. Fundamental theorem of symmetric polynomials implies that s is a polynomial in the elementary symmetric polynomials. When we wanted to evaluate $r_1, \dots, r_n$, it can be seen that $s(r_1, \dots, r_n)$ is in elementary symmetric polynomial.

□

Now, if we wanted to have some idea on how this corollary works.

Example 6.2. *Suppose that $f = x^3 + 2x^7 + x + 7 \in \mathbb{Q}$, considering this it has roots from $r_1, \dots r_n$. Again we let s to be another polynomial that is monic with roots $r_1 + r_2, r_1 + r_3$ and $r_2 + r_3$, from here we can claim that s has to have coefficients in $\mathbb{Q}$. In order to prove this we can proceed as follows,*

$$s(x) = (x - (r_1 + r_2))(x -_(r_1 + r_3)(x - (r_2 + r_3)) \tag{46}$$
$$= x^3 - (2r_1 + 2r_2 + 2r_3)x^2 + (r_1^2 + r_2^2 + r_3^2 + 3r_1r_2 + 3r_1r_+3r_2r_3)x. \tag{47}$$
$$- (r_1 + r_2)(r_2 + r_3)(r_1 + r_3). \tag{48}$$

Since r_i are the roots of a polynomial with coefficients in $\mathbb{Q}$. Corollary(6.1) strongly helps us at claiming coefficients of s are in complex numbers. So $s \in \mathbb{Q}$.

In more general if $f = x^3 + bx^2 + cx + d$, has roots from $r_1, \dots r_n$ and again s is the polynomial with roots from $r_1 + r_2, r_2 + r_3, r_1 + r_3$, we showed this (46)-(48). So $s(x)$ is,

$$s(x) = x^3 + 2bx^2 + (b^2 + c)x + bc - d. \tag{49}$$

7 Uniqueness

We have proved FTSP. Which implied that every symmetric polynomials can be written in terms of elementary symmetric polynomials. Now we will consider this and try to prove that this expression is unique. So the new theorem would be as followed.

Theorem 7.1. Every symmetric polynomial from $x_1, \dots, x_n$ can be written in terms of elementary symmetric polynomials in $\sigma_1, \dots, \sigma_n$, in only one way.(FTSP unique for each symmetric polynomial).

Proof. Recalling the polynomial ring $F[u_1, \dots, u_n]$, where u_i's are our new variables. Recall our our theorem(2.5), yea seems useful. That mapping u_i to $\sigma_i\ inF[x_1, \dots, x_n]$ defines a ring homomorphism,

$$\varphi : F[u_1, \dots, u_n] \to F[x_1, \dots, x_n]. \tag{50}$$

Meaning that $s = s(u_1, \dots, u_n)$ is a polynomial in $u_1, \dots, u_n$ with coefficients in the same field F, so $\varphi(s) = s(\sigma_1, \dots, \sigma_n)$.

Image of φ, is the set of all polynomials in the elementary symmetric polynomials. This image can be shown as follows,

$$F[\sigma_1, \dots, \sigma_n] \subset F[x_1, \dots, x_n]. \tag{51}$$

So, $F[\sigma_1, \dots, \sigma_n]$ is subring of $F[x_1, \dots, x_n]$. Now φ can be written as a map.

$$\varphi : F[u_1, \dots, u_n] \to F[x_1, \dots, x_n]. \tag{52}$$

This map must be surjective since it is elementary symmetric polynomial. Uniqueness of it will be proved by showing that mapping is one to one.

In order to prove that it is one-to-one, it is enough to mention that its kernel is 0, meaning the inverse image. So we must show if s is nonzero polynomial in u_i then, $s(\sigma_1, \dots, \sigma_n)$ will give us a nonzero polynomial. David cox does not give the rest of the proof but he wants us to do it, he leaves us it as exercise in 3 sub problems. He actually gave us some hint on how to prove it since he put them to the questions, which are necessary steps to complete the proof. And now we will proceed as followed.

So the remaining goal is to prove that $s(\sigma_1, \dots, \sigma_n)$ is a nonzero polynomial from $x_1, \dots, x_n$.

a) If $cu_1^{b_1} \dots u_n^{b_n}$ is a term of s, then show that leading term of $c\sigma_1^{b_1} \dots \sigma_n^{b_n}$ is $cx_1^{b_1+\cdots+b_n} x_1^{b_2+\cdots+b_n} \dots x_n^{b_n}$.

b) Show that $(b_1, \dots, b_n) \mapsto (b_1 + \cdots + b_n, b_2, \dots, b_n, \dots, b_n)$, is one to one.

c) To see why $s(\sigma_1, \dots, \sigma_n)$ is nonzero, consider the term of $s(u_1, \dots, u_n)$ for which the leading term of $c\sigma_1^{b_1} \dots \sigma_n^{b_n}$ is maximal. Prove that this leading term is in fact the leading term of $s(\sigma_1, \dots, \sigma_n)$, and explain how this proves what we want.

Proof. a) The leading term of a product is actually the product of the leading term of the factors, and the leading term of σ_r is $x_1 \dots x_n$, then leading term of $c\sigma_1^{b_1}\sigma_n^{b_n} \dots \sigma_n^{b_n}$ is

$$(c\sigma_1^{b_1} \dots \sigma_n^{b_n}) = c(x_1)^{b_1}(x_1x_2)^{b_2} \dots (x_1x_2 \dots x_n)^{b_n}. \tag{53}$$

$$= cx_1^{b_1+\cdots+b_n} x_2^{b_2+\cdots+b_n} \dots x_n^{b_n}. \tag{54}$$

b) Now consider sum of $b_1 + \cdots + b_n$ and so on $b_2 + \cdots + b_n$, and so on. Generalize those sum as k_i. then we will have $b_1 = k_1 - k_2$ since only differences between those sums is b_1 because k_1 starts from b_1. Same thing for $b_2 = k_2 - k_3$. So this is from $f : \mathbb{Z} \to \mathbb{Z}$ is defined by the statement that we try to prove which is now can easily seen that bijective.

c) If $s \neq 0$ there must be a term $cu_1^{b_1} \ldots u_n^{b_n}$ of s so that leading term of $cx_1^{a_1} \ldots x_n^{a_n}$ of $c\sigma_1^{b_1} \ldots \sigma_n^{b_n}$ is maximal as we said before. And every other term $c^{'}u_1^{b_1^{'}}u_2^{b_2^{'}} \ldots u_n^{b_n^{'}}$ of s is correcting $(b_!^{'}, \ldots, b_n^{'}) \neq (b_1, b_2, \ldots b_n)$ and the leading term $c^{'}x_1^{a_1^{'}} \ldots x_n^{a_n^{'}}$ of $c^{'}u_1^{b_1^{'}}u_2^{b_2^{'}} \ldots u_n^{b_n^{'}}$ is less than $cx_1^{a_1} \ldots x_n^{a_n}$. I mean we know it is not greater and equal to because $(a_1, \ldots a_n) \neq$, we know it from (b) is bijective. We also previously mentioned a lot from lexicographic order and graded one as well. With regard to that, monomials being total order, $x_1^{a_1} \ldots x_n^{a_n} > x_1^{a_1^{'}} \ldots x_n^{a_n^{'}}$. Same thing for $cx_1^{a_1} \ldots x_n^{a_n}$ which is greater than the leading terms of every other $c\sigma_1^{a_1^{'}} \ldots \sigma_n^{a_n^{'}}$ of $s(\sigma_1, \ldots, \sigma_n) \neq 0$, so greater than every other term of that. In the end, I know that was long, they don't cancel and we left with $s(\sigma_1, \ldots, \sigma_n)$.

□

□

We will talk more about symmetric polynomials with different definitions and different applications in the section of Schur functions.

8 About Symmetric Polynomials

Since symmetric polynomials has studies by many mathematicians including Euler and Waring, but Albert Girard was was one of the first one who studied on them. He showed a clear definition of elementary symmetric polynomials in his book *Invention Nouvelle en l'algebre*. He also gave some formulas for power sums in terms of elementary symmetric polynomials. Now on Newton, yes famous one and somehow he is in everywhere, I had no clue before I dive into symmetric polynomials that he studies them. In his book of *Arithmetica Universalis* had parts focused on how elementary symmetric polynomials were related to power sums. His identities were also pretty much related to symmetric polynomials. Waring has studied the topic as well, in his book *Meditationes Algebraice*, he has used implicit algorithm of what we did. FTSP were often used by 18th century, even though first formal has done by Gauss in 19th century. The proof David cox used his proof as well, and Gauss was also the first one who take the "uniqueness" to the scene. We owe him a lot

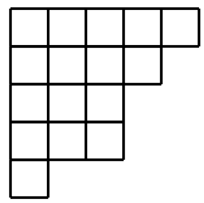
9 Schur Functions

Schur functions first studies by, German mathematician Carl Gustav Jacob Jacobi, as skew symmetric polynomial by a_s, just as we showed in the last chapter. Their connection to representation which I come across lots of time while learning the topic, was discovered by Issai Schur. Schur functions or Schur polynomials are expelled from many different topics. These polynomials are actually and there are many ways to define them, which is needed for different applications of Schur Functions.

9.1 Tableaux

Definition 9.1. Partition $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$,($\vdash$ has many usages but in here we used it for meaning that "λ is a partition of the integer n", which is the usage that used in combinatorics), in here s must be greater than 1. Introducing Ferrers diagram which is used to represent the partition λ as dots in determined rows. And ith row contains i number of squares.

Example 9.2. *Recalling partitions* $\lambda = (5, 4, 3, 3, 1)$, *in Ferrers diagram its representation is as follows,*



(55)

To make more useful diagrams sometimes or often we might need to fill inside the tableaux. It is defined as followed.

Definition 9.3. Let $\mu = (\mu_1, \ldots, \mu_n k)$ is denoted as filling diagrams. And μ_j appears j times.

Example 9.4. *With regard to definition* 9.3, *we can make a Ferrers diagram with* $\lambda = (5, 3, 3, 2), \mu = (4, 4, 5)$. *Which is illustrated as,*

1	2	1	2	1
1	2	2		
3	3	3		
3	3			

This is a completely filled Ferrers diagram, for now it does not matter how the numbers have distributed.

9.2 Alternating Polynomials

Definition 9.5. Multi-index is used for shortening expression where lots of variables contained. A multi-index α is an n-tuple of integers a_i, where i starts from 1 to n.$\alpha = (\alpha - 1, \ldots, \alpha_n)$. $|\alpha|$ is sum of all a_i.

Definition 9.6. Let finite variables $x_1, \ldots, x_n$, and alternating polynomials is form of,

$$f(x_1, \ldots, x_n) = \sum_{a=1}^{n} c_a x_a, \tag{56}$$

$c(a_{\omega(1)}, \ldots, a_{\omega(n)}) = \epsilon(\omega) c(a_1, \ldots, a_n)$ for all multi-index, with all permutation in symmetric group S_n. ϵ denotes the sign function, where it just returns sign of a real number either $1, -1$. In alternating polynomials signs are reversed upon changing the order of any variables.

Lemma 9.7. For any multi-index if $\alpha_i = \alpha_j$ where they i and j are not equal, then we get $c_\alpha = 0$. All monomials in the form of alternating polynomial must be made with distinct x. Implies to that polynomial will be made with decreasing multi-indices, so it makes sense.

Definition 9.8. Again beginning with variables $x_1, \ldots, x_n$. Letting $x^\alpha = x_1^{a_1}, \ldots x_n^{a_n}$ be a monomial, and consider the polynomial a_α is obtained by antisymmetryizing, meaning that,

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{\omega \in S_n} \epsilon(\omega)\omega(x^\alpha) \tag{57}$$

$\epsilon(\omega)$ is sign either 1 or -1 of all permutation ω, and a_α is called skew symmetric. With all those,

$$\omega(a_\alpha) = \epsilon(\omega)a_\alpha. \tag{58}$$

For all ω in the symmetric group. Therefore a_α disappear till all a's are different than any other. We can also assume that from 1 to n they are in decreasing order. So it can be written as $a = \lambda + \delta$. Where δ is from $n-1$ to 0, where $\lambda \leq n$. Then we will get,

$$a_\alpha = a_{\lambda\delta} = \sum_\omega \epsilon(\omega)\omega(x^{\lambda+\delta}), \tag{59}$$

which is can also be interpreted as determinant,

$$a_{\alpha+\delta} = det(x_i^{\delta_j+n-j}). \tag{60}$$

This is divisible by $x_i - x_j$ where i is smaller than j, hence with the product of them we get,

$$\Pi_{1\leq i<j\leq n}(x_i - x_j) = det(x_i^{n-j}) = a_\delta. \tag{61}$$

From here we conclude that $a_{\delta+\lambda}$ can be divided by a_δ which is in $\mathbb{Z}[x_1, \ldots, x_n]$. So quotient is,

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = \frac{a_{\delta+\lambda}}{a_\delta}. \tag{62}$$

This is symmetric as well then, which is called **Schur Function** with the variables from $x_1, \ldots, x_n$. This adds up to partition λ and homogeneous since variables by some power.

Mentioned earlier but now we will look at some other definitions of Schur Functions. Which might be better to have the idea where an example of it will be illustrated.

Definition 9.9. For fix λ, bound N the size of the entries in all semistandard tableau T. A tableau is semistandard if entries weakly increase along each row and strictly increase down in each column. Let $x^T = \Pi_{i=1}^N x_i^j$, where j represents number of i's in tableau T. From here Schur Polynomial defined as $s_\delta(x_1, \ldots, x_n) := \sum_{semistandard} x^T$.

Example 9.10. *It is easier to see with grids. Let $\delta = (2,1)$. Then the list of possible tableaux in the shape of δ where $N = 3$ are,*

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

Now, we have given definition for Schur Polynomial, it not okay till we give the corresponding polynomial.

$$s_{(2,1)(d_1,x_2,x_3)=x_1^2x_2+x_1^2x_3+x_1x_2^2+x_1x_2x_3} \tag{63}$$

10 Bibliography

References

[1] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms, Springer-Verlag, New York, 1997

[2] R. Ab lamowicz, B. Fauser, SP - A Maple Package for Symmetric Polynomials, http://math.tntech.edu/rafal/, April 2009

[3] I. G. Macdonald. Symmetric Functions and Hall Polynomials. Oxford University Press, second edition, 1995.

[4] L. Manivel. Symmetric Functions, Schubert Polynomials and Degeneracy Loci. AMS/SMF, 1998.

[5] W. Fulton, Young Tableaux, Cambridge University Press, 1997