Geometric Inequalities and Optimizations

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Isoperimetric Problem for Triangles

Theorem

Given a fixed perimeter p, the triangle with maximum area with perimeter p is an equilateral triangle.

Proof

Let $p = a + b + c$, where a, b, and c are the side lengths of the triangle. By Heron's formula, we have that

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter of the triangle. Rewriting the area in terms of a, b, and c gives

$$A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)}.$$

Proof

Now assume without loss of generality that $a > b$. Pick ϵ such that $a - \epsilon \geq b + \epsilon$. We let $a' = a - \epsilon$ and $b' = b + \epsilon$, and we plug a' in for a and b' in for b. The first two terms under the square root remain the same, but the last two change. The last two become (ignoring the 2s in the denominators)

$$(a - b - 2\epsilon + c)(-a - b + 2\epsilon + c) = -(a - b - 2\epsilon)^2 + c^2.$$

Proof

Comparing this to the expansion of the orginial last two terms, $-(a-b)^2 + c^2$, we see that the expression has grown larger. Thus, when we push two of the variables closer together, we get that the area increases. Therefore, the maximum area is achieved when all the variables are equal, or when the triangle is equilateral.

Isoperimetric Problem for n-gons

Theorem

Given a fixed perimeter p, the maximum area an n-gon with perimeter p occurs when the n-gon is regular.

Isoperimetric Problem for n-gons

We use a similar strategy as we did for triangles in the sense that we push two objects closer together in value and show that area increases.

Proof

First note that if S is not convex, we can make it convex with the same perimeter. Find a nonconvex portion of the polygon and reflect over a the line connecting the two end vertices of this nonconvex portion (word better). Thus, S must be convex.

Proof

Now consider two consectuve sides of S. Let the endpoints of these sides be A, B, and C. We claim the area of $\triangle ABC$ is maximized when $AB = AC$. Note that B lies on an ellipse with foci at A and C. Varying B on this ellipse lets you increase the area while keeping the perimeter fixed.

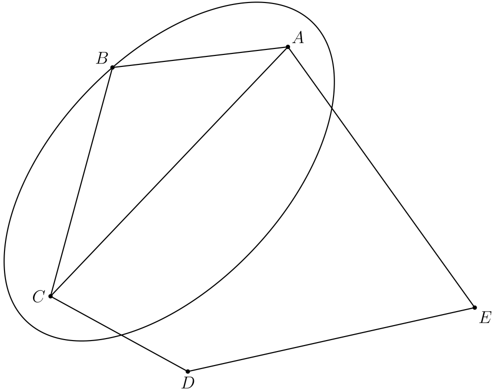

Figure: Example for a pentagon

Proof

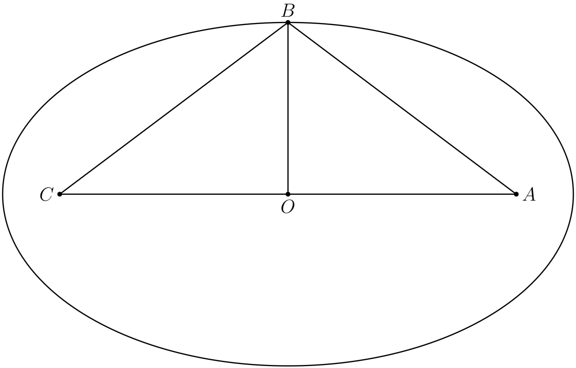

Figure: Configuration with max area

Proof

Now consider three consecutive sides, with vertices A, B, C, and D. We know that $AB = BC = CD$. We claim the area is maximized when the angles $\angle ABC = \angle BCD$.

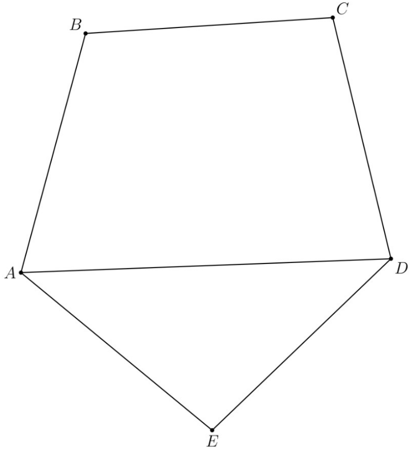

Figure: Example for pentagon, all sides are equal

Proof

Theorem (Bretschneider's Formula)

The area of quadrilateral ABCD with side lengths a, b, c, and d, semiperimeter s, and opposite angles α and γ is

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)}.$$

Proof

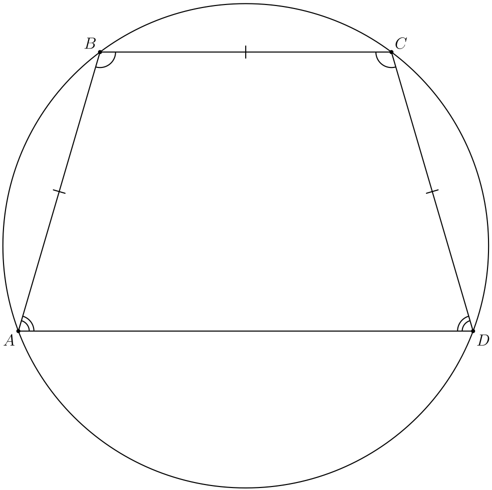

Figure: Configuration with max area

Proof

We've established that maximum area occurs when S is convex, S is equilateral, and S is equiangular. Therefore, S has maximum area when S is regular, as desired.

Corollary

The maximum area of an n-gon with perimeter p is $\frac{p^2/n}{4\tan(180/n)}$.

Proof

Decompose the regular n-gon (it has the max area) into n congruent isoceles triangles, with the base vertices as two consecutive vertices on the n-gon and the other vertex being the center of the n-gon. The vertex angle is $\frac{360}{n}$, and the other two angles are $90^\circ - \frac{180}{n}$. We know that the side opposite of the vertex is $\frac{p}{n}$. The area of a triangle with all angles and one side is $A = \frac{a^2 \sin B \sin C}{2 \sin A}$, where a is opposite A.

Proof

Plugging in our values, we obtain that the area of one of these triangles is

$$A = \frac{(p/n)^2 \sin^2\left(90^\circ - \frac{180}{n}\right)}{2\sin\left(\frac{360}{n}\right)}$$
$$= \frac{(p/n)^2 \cos^2\left(\frac{180}{n}\right)}{4\sin\left(\frac{180}{n}\right)\cos\left(\frac{180}{n}\right)}$$
$$= \frac{(p/n)^2}{4\tan(180/n)}.$$

Multiplying this by n for the n triangles gives us the desired area.

Isoperimetric Inequality

Theorem (Isoperimetric Inequality)

Let l be a fixed length. For any closed curve with perimeter l, the area A of the curve satisfies this inequality:

$$\frac{l^2}{4\pi} \geq A,$$

with equality occuring when the curve is a circle.

Lemma 1

Given a closed curve C, let $c(t) = (x(t), y(t))$ be its parametrization, with $t \in [a, b]$ and $c(a) = c(b)$. Let the area bounded by the curve be A. Then,

$$A = \int_a^b xy' \, dt = -\int_a^b x'y \, dt.$$

Lemma 1

Proof: By Green's Theorem, we have that

$$A = \oint_C xy' = -\oint_C x'y.$$

Evaluating these gives the desired conclusion.

Lemma 2

Let x, y, and z be functions of t with continuous first derivatives. We have that

$$(xy' - zx')^2 \leq (x^2 + z^2)((x')^2 + (y')^2)$$

Lemma 2

Proof: By the Trivial Inequality,

$$\begin{aligned} 0 &\leq (xx' + zy')^2 \\ &= x^2(x')^2 + 2xx'zy' + z^2(y')^2 \\ &= x^2(x')^2 + x^2(y')^2 + z^2(x')^2 + z^2(y')^2 - (x^2(y')^2 - 2xy'zx' + z^2(x')^2) \\ &= (x^2 + z^2)((x')^2 + (y')^2) - (xy' - zx')^2. \end{aligned}$$

Equality occurs when $xx' = -zy'$.

Proof

Now we begin the proof. Let $c(t) = (x(t), y(t))$ be the perimetrization of the positively oriented closed curve C with length l. Let $I = [-r, r]$ such that $x(t) \in I$ (graphically, these are two parallel lines tangent to C such that C is entirely between them). Without loss of generality, let $x(0) = x(l) = r$ and $x(m) = -r$ for some $0 < m < l$. Define $k(t) = (x(t), z(t))$ to be a circle with radius r ($z(t) = \sqrt{r^2 - x(t)^2}$ for $0 < t \leq m$ and $z(t) = -\sqrt{r^2 - x(t)^2}$ for $m < t \leq l$).

Proof

Let A be the area of C and B the area of the circle. By Lemma 1, we have that

$$A = \int_0^l x(t)y'(t)\,dt$$

and

$$B = -\int_0^l x'(t)z(t)\,dt = \pi r^2.$$

Proof

Adding these together gives

$$\begin{aligned} A + \pi r^2 = A + B &= \int_0^l xy' - x'z \, dt \\ &\le \int_0^l \sqrt{(xy' - x'z)^2} \, dt \\ &\le \int_0^l \sqrt{(x^2 + z^2)((x')^2 + (y')^2)} \, dt \\ &= \int_0^l r \, dt = rl, \end{aligned}$$

where the second inequality comes from Lemma 2, and the second to last equality comes from $x^2 + z^2 = r^2$ and $(x')^2 + (y')^2 = 1$, since the curve is parametrized by arc length. By AM-GM,

$$rl = A + \pi r^2 \ge 2\sqrt{A\pi r^2} \implies r^2 l^2 \ge 4A\pi r^2 \implies \frac{l^2}{4\pi} \ge A,$$

as desired.

Equality

To find out when equality occurs, we note that since we used AM-GM at the end, A has to equal πr^2, which means $l = 2\pi r$. By Lemma 2, equality between the second and third integral occurs when $-xx' = zy'$. Note that $x^2 + z^2 = r^2$ and $(x')^2 + (y')^2 = 1$. From here, we reduce the equality $(xy' - zx')^2 = (x^2 + z^2)((x')^2 + (y')^2)$ into a function of y'.

Equality

$$(xy' - zx')^2 = (x^2 + z^2)((x')^2 + (y')^2) \implies x^2(y')^2 - 2zx'xy' + z^2(x')^2 = r^2$$

$$\implies x^2(y')^2 + 2(x')^2x^2 + z^2(x')^2 = r^2$$

$$\implies x^2(y')^2 + (x')^2x^2 + (x')^2x^2 + z^2(x')^2 = r^2$$

$$\implies x^2((x')^2 + (y')^2) + (x')^2(x^2 + z^2) = r^2 \implies x^2 + (x')^2(x^2 + (r^2 - x^2))$$

$$\implies x^2 + (x')^2r^2 = r^2 \implies x^2 = r^2(1 - (x')^2) \implies x^2 = r^2(y')^2 \implies x = \pm ry'.$$

Equality

Now we show that $y = \pm rx'$. We now repeat the entire above argument from proving the inequality to achieving this equality case, except when we parametrize C, we now bound it by two parallel lines except these parallel lines will be perpendicular to the intial two parallel lines. The lines will bound the curve in an interval $I = [-r', r']$, and the curve will be parametrized as $c'(t) = (w(t), y(t))$, where $y(t)$ is the same as above. Repeating the procedure we get that $A = \pi(r')^2$, but A has the same area in both cases, so $r' = r$. Similarly, $-xw' = yy'$, so after the equality calculation, we get that $y \pm rx'$.

Equality

Finally, squaring both equalities with x and y and adding them gives

$$x^2 + y^2 = r^2((x')^2 + (y')^2) = r^2,$$

which is a circle, so we are done.