Geometric Inequalities and Optimizations

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Contents

Abstract

In this paper, we explore some well-known geometric inequalities, along with a class of problems known as isoperimetric problems, which deal with finding the maximum area of a figure given a fixed perimeter. We hope to shed some light on a topic at the intersection of geometry and algebra, along with providing some very nice inequalities to consider.

Acknowledgements

Thank you to Abhy Devalapura for advice during the creation of this paper.

Part I. Inequalities on Triangles

1 Triangle Inequality

One of the foundations of Euclidean geometry is the following seemingly obvious fact: the shortest distance between two points in the plane is the straight line segment between them. In fact, this observation does not hold on different geometric sets (i.e. the shortest distance between two points on a sphere is not a straight line). So, how does one go about proving the fact rigorously? We resort to using analytic geometry.

Theorem. *The shortest distance between two points in the plane is the straight line segment between them.*

Proof: Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x, y)$. We want to show that the shortest distance between A and B occurs when we use the straight line distance between them, and not when two segments are involved. So, what we want to show is that $AB \leq AC + BC$. First, we note that we can shift and rotate the points so that C becomes the origin (these two preserve distance). Let our points now be $A = (a_1, b_1)$, $B = (a_2, b_2)$, and $C = (0, 0)$. Using the distance formulas, we obtain that this is equivalent to

$$\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \leq \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2}.$$

Squaring both sides and rearranging gives

$$-2a_1a_2 - 2b_1b_2 \leq 2\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}.$$

Squaring this and dividing by 4 gives

$$(a_1a_2 + b_1b_2)^2 \leq (a_1^2 + b_1^2)(a_2^2 + b_2^2).$$

For those knowledgable about inequalities, it is easy to note that is the Cauchy-Schwarz inequality when $n = 2$.

Now, we live in a 3-dimensional space. In our world, it also seems fairly obvious that the shortest distance between two points is a straight line (ignoring the fact that we live on a curved surface). So, how would we extend the triangle inequality to 3 dimensions? We can simply do the same thing we did in 2D. Let $A = (a_1, b_1, c_1)$, $B = (a_2, b_2, c_2)$, and $C = (0, 0, 0)$. By distance formulas, we get that

$$\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2} \leq \sqrt{a_1^2 + b_1^2 + c_1^2} + \sqrt{a_2^2 + b_2^2 + c_2^2}.$$

Expanding this and doing some rearranging will gives us that the inequality is equivalent to

$$(a_1a_2 + b_1b_2 + c_1c_2)^2 \le (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2).$$

This is just Cauchy-Schwarz for $n = 3$. In fact, the Triangle Inequality in k dimensions just reduces to Cauchy-Schwarz for $n = k$. Therefore, the triangle inequality is true in all n-dimensional spaces for $n \ge 2$. Since Cauchy-Schwarz is always true, we do indeed have the original inequality being true. Note that the equality case of Cauchy-Schwarz occurs when $a_1{:}b_1{:}c_1,\ a_2{:}b_2{:}c_2, \dots a_n{:}b_n{:}c_n$, which implies that A, B, and C are collinear when equality occurs. □

The importance of this theorem cannot be understated. Immediately from it, we obtain one of the most fundamental properties of triangles.

Corollary (Triangle Inequality)**.** *In a non-degenerate $\triangle ABC$, let a, b, and c be the sidelengths of BC, CA, and AB respectively. Then,*

$$a > b + c,$$
$$b > a + c,$$
$$c > a + b.$$

Proof: Since A does not lie on $\overline{BC}$, the first inequality follows from the theorem. The other inequalities follow similarly. □

There exist generalizations generalizations of the triangle inequality. We state one for complex numbers without proof. For complex numbers z_1, z_2, $\dots$ z_n,

$$\sum_{i=1}^{n} |z_i| \ge \left|\sum_{i=1}^{n} z_i\right|.$$

2 Erdos-Mordell Inequality

Theorem (Erdos-Mordell Inequality)**.** *Let P be a point on the interior of $\triangle ABC$. Let D, E, and F be the foots from P to BC, CA, and AB respectively. Then,*

$$PA + PB + PC \ge 2(PD + PE + PF),$$

with equality holding when $\triangle ABC$ is equilateral and P is the center of $\triangle ABC$.

Proof: Let M and N be the projections of E and F onto line PD. Note that PA is the diameter of cyclic quadrilateral $AFPE$, so by law of sines,

$$\frac{EF}{\sin A} = 2R = PA \implies EF = PA \sin A.$$

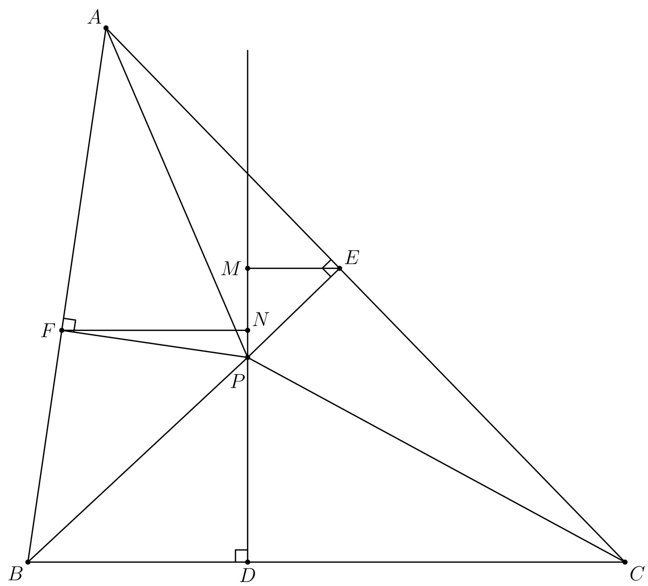

Fig. 1: FN and ME are perpendicular to PD

Note that $BDPF$ is cyclic, so $\angle B = \angle FPN$. This means that

$$\frac{FN}{FP} = \sin FPN = \sin B \implies FN = PF \sin B.$$

Similarly, $EM = PE \sin C$. Note that $EF \geq FN + EM$ by Pythogorean Theorem, so $PA \sin A \geq PE \sin C + PF \sin B \implies PA \geq PE \cdot \frac{\sin C}{\sin A} + PF \cdot \frac{\sin B}{\sin A}$. Doing this same process on quadrilaterals $FPBD$ and $EPDC$, we get the following set of inequalities:

$$PA \geq PE \cdot \frac{\sin C}{\sin A} + PF \cdot \frac{\sin B}{\sin A},$$
$$PB \geq PF \cdot \frac{\sin A}{\sin B} + PD \cdot \frac{\sin C}{\sin B},$$
$$PC \geq PD \cdot \frac{\sin B}{\sin C} + PE \cdot \frac{\sin A}{\sin C}.$$

Adding them all gives

$$PA+PB+PC \geq PD\left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right) + PE\left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}\right) + PF\left(\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A}\right).$$

Applying AM-GM on $\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}$ and cyclic variants, we get that

$$PA + PB + PC = 2(PD + PE + PF).$$

$\square$

3 Ciamberlini's Inequality

Theorem (Ciamberlini's Inequality)**.** *In non-obtuse triangles, we have that*

$$s \geq 2R + r,$$

with equality when the triangle is right-angled. In obtuse triangles, we have that

$$s < 2R + r.$$

To prove this, we will prove the following property of triangles:

Theorem. *In any triangle, we have that*

$$\frac{s^2 - (2R + r)^2}{4R^2} = \cos A \cos B \cos C.$$

Proof: We will use the well-known fact that $1 + \frac{r}{R} = \cos A + \cos B + \cos C$.

$$\begin{aligned}
\frac{s^2 - (2R + r)^2}{4R^2} &= \frac{1}{4}\left(\frac{s}{R}\right)^2 - \frac{1}{4}\left(\frac{2R + r}{R}\right)^2 \\
&= \frac{1}{4}\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)^2 - \frac{1}{4}\left(2 + \frac{r}{R}\right)^2 \\
&= \frac{1}{4}[(\sin A + \sin B + \sin C)^2 - (1 + \cos A + \cos B + \cos B)^2] \\
&= \frac{1}{4}[(\sin^2 A + \sin^2 B + \sin^2 C + 2\sin A \sin B + 2\sin B \sin C + 2\sin C \sin A) \\
&\quad - (1 + 2\cos A + 2\cos B + 2\cos C + 2\cos A \cos B \\
&\quad + 2\cos B \cos C + 2\cos C \cos A + \cos^2 A + \cos^2 B + \cos^2 C)] \\
&= \frac{1}{4}(-\cos 2A - \cos 2B - \cos 2C + 2(\sin A \sin B - \cos A \cos B + \sin B \sin C \\
&\quad - \cos B \cos C + \sin C \sin A - \cos C \cos A - \cos A - \cos B - \cos C) - 1) \\
&= \frac{1}{4}[-\cos 2A - \cos 2B - \cos 2C + 2(-\cos(A + B) - \cos(B + C) \\
&\quad - \cos(C + A) - \cos A - \cos B - \cos C) - 1] \\
&= \frac{1}{4}[-2\cos(A + B)\cos(A - B) - 2\cos^2 C + 1 \\
&\quad + 2(\cos C + \cos A + \cos B - \cos A - \cos B - \cos C) - 1] \\
&= -\frac{1}{2}(-\cos C \cos(A - B) + \cos^2 C) \\
&= \frac{1}{2}[\cos C(-\cos C + \cos(A - B))] \\
&= \frac{\cos C}{2}(\cos(A + B) + \cos(A - B)) \\
&= \cos A \cos B \cos C.
\end{aligned}$$

From this, we get that $s^2 = (2R+r)^2 + 4R^2 \cos A \cos B \cos C$. Note that when the triangle is acute, $(2R+r)^2 + 4R^2 \cos A \cos B \cos C > (2R+r)^2$, so $s > 2R+r$. When the triangle is right-angles, $(2R+r)^2 + 4R^2 \cos A \cos B \cos C = (2R+r)^2$, so $s = 2R+r$. when the triangle is obtuse, $(2R+r)^2 + 4R^2 \cos A \cos B \cos C < (2R+r)^2$, so $s < 2R+r$. □

There exist many more inequalities that relate these quantities in a triangle. One can find more about these inequalities here [1].

Part II. Inequalities on Quadrilaterals

In this section, we look at two inequalities that are based on the lenghts of a quadrilateral.

4 Ptolemy's Inequality

Theorem (Ptolemy's Inequality)**.** *For any quadrilateral $ABCD$, the following inequality holds, and equality occurs when the four points are concyclic:*

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD.$$

Proof: Let E be the point such that $\triangle ACD \sim \triangle AEB$. From this, we have that

$$\frac{AC}{AE} = \frac{CD}{EB} = \frac{DA}{BA} \implies EB = \frac{BA \cdot CD}{DA}.$$

Now note that $\angle EAC = \angle AEB + \angle BAC = \angle CAD + \angle BAC = BAD$ and $\frac{DA}{AC} = \frac{BA}{AE}$. This implies that $\triangle EAC \sim \triangle BAD$, which further implies that

$$\frac{CE}{AC} = \frac{BD}{AD} \implies CE = \frac{BD \cdot AC}{AD}.$$

Now, from the triangle inequality we get that $CE \leq EB + BC$. Substituting our expressions for EB and CE, we get that

$$\frac{BD \cdot AC}{AD} \leq EB = \frac{BA \cdot CD}{DA} + BC \implies AB \cdot CD + BC \cdot CA \leq AC \cdot BD,$$

which is what we wanted to prove. Note that equality occurs when $CE = EB + BC$, meaning B, C, and E are collinear. This implies that $\angle ADC = \angle ABE = 180^\circ - \angle ABC$, where the first equality comes from similar triangles. This means equality occurs when $ABCD$ is cyclic. □

5 Euler-Pythagoras Inequality

Theorem (Euler-Pythagoras Theorem)**.** *In a quadrilateral $ABCD$, we have the following inequlity, with equality when $ABCD$ is a parallelogram:*

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2.$$

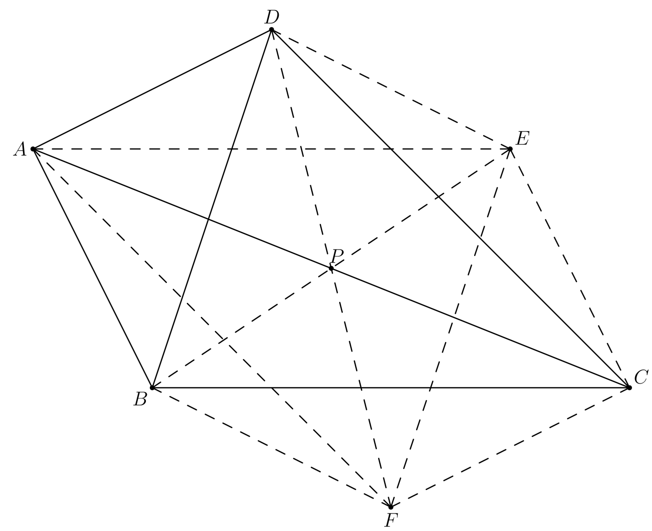

Fig. 2: P bisects AC, DF, and BE

Proof: Let P be the midpoint of AC. Let E be the point such that $ABCE$ is a parallelogram, and F be the point such that $ADCF$ is a parallelogram. Note that since $ADCF$ is a parallelogram, DF and AC bisect each other at P. Similarly, BE and AC bisect each other at P. But this implies that DF and BE bisect each other at P, so $BDEF$ is a parallelogram.

Note that

$$\begin{aligned}
\angle DAC &= \angle ACF \\
\angle DAE + \angle EAC &= \angle ACB + \angle BCF \\
\angle DAE + \angle EAC &= \angle EAC + \angle BCF \\
\angle DAE &= \angle BCF.
\end{aligned}$$

Similarly, $\angle DEA = \angle CBF$, and since $BC = AE$, $\triangle DEA \cong \triangle BCF$.

Consider an arbitrary parallelogram $ABCD$. By law of cosines, we have that

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cos \angle ABC \tag{1}$$

and

$$BD^2 = AB^2 + AD^2 - 2 \cdot AB \cdot AD \cos \angle BAD. \tag{2}$$

We have that $AD = BC$ and $\cos \angle ABC = -\cos \angle BAD$, so we can rewrite (2) as

$$BD^2 = AB^2 + BC^2 + 2 \cdot AB \cdot BC \cos \angle ABC. \tag{3}$$

Adding (1) and (3) we get that

$$AC^2 + BD^2 = 2AB^2 + 2BC^2.$$

Applying this to parallelogram $ADCF$, $BDEF$, and $AECB$, we obtain that

$$\begin{aligned} 2AD^2 + 2CD^2 &= AC^2 + DF^2, \\ 2BD^2 + 2DE^2 &= BE^2 + DF^2, \\ 2AB^2 + 2BC^2 &= AC^2 + BE^2. \end{aligned}$$

Subtracting the first two equations and rearranging gives

$$2AD^2 + 2CD^2 = 2BD^2 + 2DE^2 + AC^2 - BE^2.$$

Adding the third eqaution to this and dividing by 2 gives

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + DE^2,$$

which implies our desired inequality. Note that DE is equal to 0 when $ABCD$ is already a parallelogram, so that's when equality occurs. □

Part III. Inequalities on Circles

6 Euler's Inequality

This inequality relates the inradius and circumradius, and is a nice application of power of a point and similar triangles.

Theorem (Euler's Inequality)**.** *Let* $\triangle ABC$ *have inradius* r *and circumradius* R*. Then*

$$R \geq 2r$$

with equality when $\triangle ABC$ *is equilateral.*

This theorem is a direct result of the following theorem, which is what we will prove first:

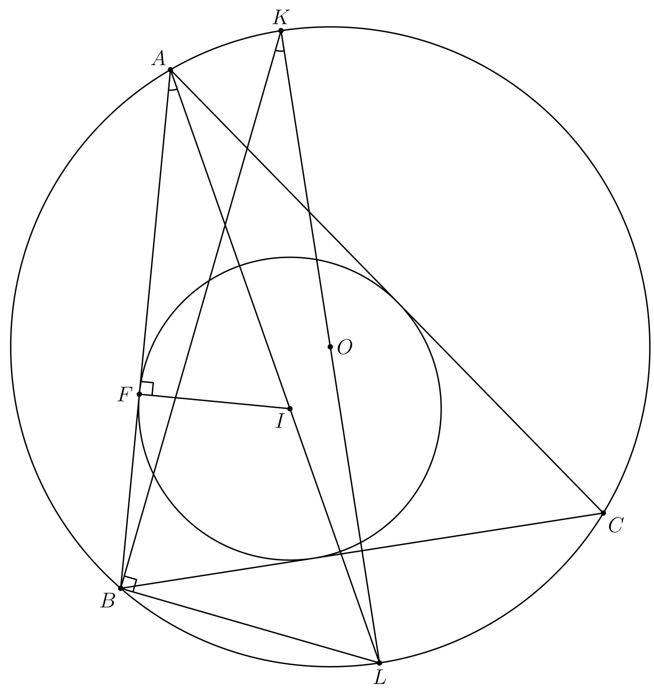

Fig. 3: $\triangle AFI \sim \triangle KBL$

Theorem. *Let $\triangle$ ABC have inradius r and circumradius R, incenter I and circumcenter O. Then we have that*

$$OI^2 = R(R - 2r).$$

Proof: Denote (ABC) and ω. Rewriting the condition, we get that

$$R^2 - OI^2 = 2Rr.$$

The left is exactly the power of I with respect to ω. Let ray AI intersect ω at L. By power of a point,

$$AI \cdot IL = R^2 - OI^2 = 2Rr,$$

so it suffices to prove thar $AI \cdot IL = 2Rr$. Rewriting the condition gives $\frac{AI}{r} = \frac{2R}{IL}$.

Let F be the foot of the perpendicular from I to side AB. Note that the left side of our condition is the ratio $\frac{AI}{IF}$. Now draw the diameter OL and let it intersect ω and $K \neq L$. Note that $KL = 2R$, and by the incenter-excenter lemma, $BL = IL$, so the ratio $\frac{KL}{BL}$ is exactly the ratio on the right side of our condition. Finally, note that $\angle AFI = \angle KBL = 90°$, and $\angle BAL = \angle BKL$ by cyclic quadrilaterals. Thus, $\triangle AFI \sim \triangle KBL$, giving

$$\frac{AI}{FI} = \frac{AI}{r} = \frac{2R}{IL} = \frac{KL}{BL}.$$

□

Euler's Inequality immediately follows from the above theorem. Simply note that the left is nonnegative, and since R is positive, $R-2r$ must be nonnegative, implying the inequality. When $R = 2r$, $OI = 0$, so the circumcenter and incenter are the same point. This only occurs in an equilateral triangle.

7 QM-AM-GM-HM on a Semicircle

Using a semicircle and constructing certain lengths, one can prove two variable QM-AM-GM-HM. We will show how to do so. As reminder, the QM-AM-GM-HM inequality for two variables states that

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

Proof: Let ω be a semcircle, with a diameter who's endpoints are A and B. Let D be the center of the semicircle, and let E be the midpoint of arc $\widehat{AB}$. Let C be an arbitrary point on segment AB (Here we assume it's to the right of D for convenience). Let F be the intersection of ω and the perpendicular to AB at C. Let G be the foot of C to FD. Let $\overline{AC} = a$ and $\overline{BC} = b$.

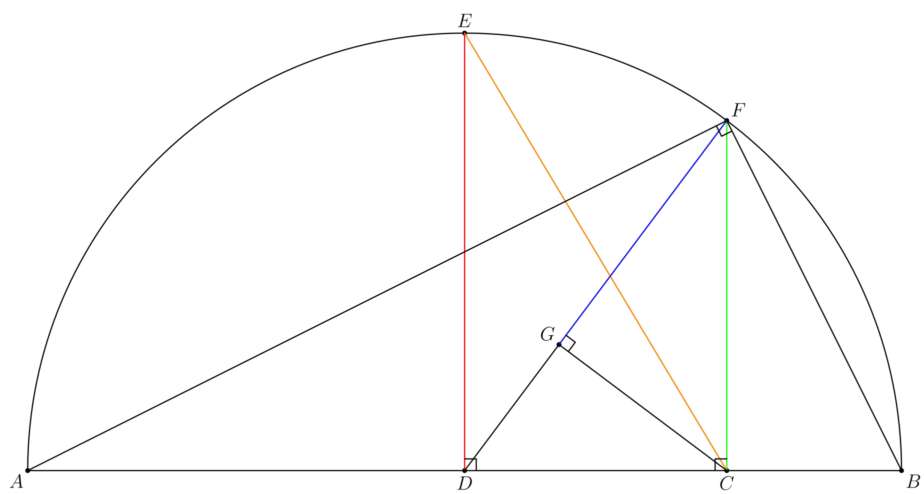

Fig. 4: Orange - QM, Red - Am, Green - GM, Blue - HM

We now go length chasing. Note that $\overline{AB} = a+b$, so the radius of ω is $\frac{a+b}{2}$. Therefore, $\overline{ED} = \frac{a+b}{2}$. We have that $\overline{DC} = \overline{AC} - \overline{AD} = a - \frac{a+b}{2} = \frac{a-b}{2}$. By Pythagorean Theorem on $\triangle EDC$,

$$\overline{EC} = \sqrt{\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2} = \sqrt{\frac{a^2+2ab+b^2}{4} + \frac{a^2-2ab+b^2}{4}} = \sqrt{\frac{a^2+b^2}{2}}.$$

By Triangle Inequality, $\overline{EC} \geq \overline{ED} + \overline{DC}$, so $\overline{EC} \geq \overline{ED} \implies \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$, which established QM-AM.

Note that $\triangle ACF \sim \triangle FCB$ ($\angle FBC = 90^\circ - \angle CFB = \angle AFC$), so $\frac{BC}{FC} = \frac{FC}{AC} \implies BC \cdot AC = FC^2 \implies \sqrt{ab} = \overline{FC}$. Since $FC \leq ED$, we have that $\sqrt{ab} \leq \frac{a+b}{2}$, establishing AM-GM.

Note that $\triangle FGC \sim \triangle FCD$, so $\frac{FG}{FC} = \frac{FC}{FD} \implies FG = \frac{FC^2}{FD}$. Note that FD is a radius, so $\overline{FD} = \frac{a+b}{2}$. Then,

$$\overline{FG} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

By Triangle Inequality on $\triangle FGC$, $\overline{FC} \geq \overline{FG} + \overline{GC}$, so $\overline{FC} \geq \overline{FG} \implies \sqrt{ab} \geq \frac{1}{\frac{1}{a}+\frac{1}{b}}$, which establishes GM-HM. $\square$

Part IV. Isoperimetric Problems

As early as the Greeks, the question of the maximum possible area a closed curve can have given a fixed perimeter was studied. The first mathematician make serious progress into proving the answer was Jacob Steiner. Steiner was able to prove that the shape had to be convex and symmetric. However, there were some flaws in Steiner's proof, so the proof of the problem had to wait until the machinery of calculus was created, allowing for expressions of area of general planar curves. In this section, we will take a look at this problem, first for triangles, then for n-gons, and finally tackle the isoperimetric problem in its full generality.

8 Isoperimetric Problem For Triangles

Our goal is the find a triangle which maximizes area given a perimeter. Messing around with different triangle configurations, a reasonable conjecture would be that an equilateral triangle maximizes the area. This is indeed the triangle that maximizes the area. To prove this, we used a smoothing argument (we are motivated to do this since the perimeter is fixed, and we can easily use Heron's formula for the triangle area).

Theorem. *Given a fixed perimeter p, the triangle with maximum area with perimeter p is an equilateral triangle.*

Proof: Let $p = a + b + c$, where a, b, and c are the side lengths of the triangle. By Heron's formula, we have that

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter of the triangle. Rewriting the area in terms of a, b, and c gives

$$A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)}.$$

Now assume without loss of generality that $a > b$. Pick ϵ such that $a-\epsilon \geq b+\epsilon$. We let $a' = a - \epsilon$ and $b' = b + \epsilon$, and we plug a' in for a and b' in for b. The first two terms under the square root remain the same, but the last two change. The last two become (ignoring the 2s in the denominators)

$$(a - b - 2\epsilon + c)(-a - b + 2\epsilon + c) = -(a - b - 2\epsilon)^2 + c^2.$$

Comparing this to the expansion of the orginial last two terms, $-(a-b)^2+c^2$, we see that the expression has grown larger. Thus, when we push two of the variables closer together, we get that the area increases. Thus, the maximum area is achieved when all the variables are equal, or when the triangle is equilateral. □

Corollary. *The maximum area of a triangle with given perimeter p is $\frac{p^2\sqrt{3}}{36}$.*

Simply let each side have length $\frac{p}{3}$ and use the area formula for an equilateral triangle.

9 Isoperimetric Problem For n-gons

Let p be the fixed perimeter of an n-gon S, and let A be the area of S. Let's assume we've guessed that for the general isoperimetric problem, the maximum area is achieved with a circle (this is for the purpose of motivation). We want to know the maximum area for an n-gon, so it's reasonable to guess that the area is maximized wehn S is as "circular" as possible. We know that regular n-gons can be inscribed in circles, so we guess that a regular n-gon maximizes the area. We will prove that A is maximized when S is regular. To do so, we need to establish three things: S is convex, S is equilateral, and S is equiangular.

First note that if S is not convex, we can make it convex with the same perimeter. Find a nonconvex portion of the polygon and reflect over a the line connecting the two end vertices of this nonconvex portion. Thus, S must be convex.

Now consider two consectuve sides of S. Let the endpoints of these sides be A, B, and C. We claim the area of $\triangle ABC$ is maximized when $AB = AC$. Note that B lies on an ellipse with foci at A and C. Varying B on this ellipse lets you increase the area while keeping the perimeter fixed.

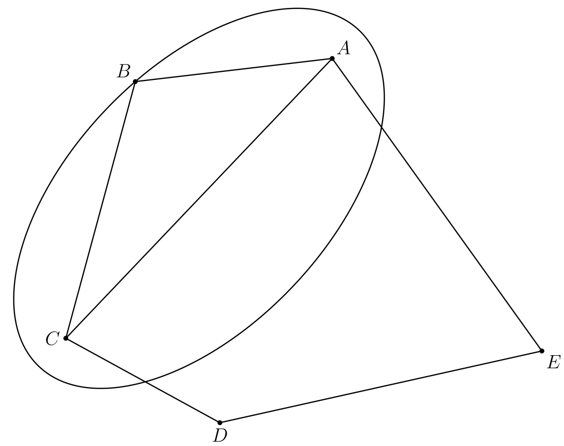

Fig. 5: Example for a pentagon

Let Γ be an ellipse centered at the origin with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, where a and b are arbitrary reals with $a > b$. We are interested in the triangle with vertices at the foci of Γ and some point on the ellipse. Since the distance between the foci is fixed, we need to find a point that maximizes the height along the ellipse. This is clearly the point on the ellipse with the maximum y value. In particular, $y = b$. Note that when we place B at this point, the side lengths BA and BC (where A and C are the foci) are the same length. Thus, when the area is maximized, the side lengths are the same length, which is what we wanted.

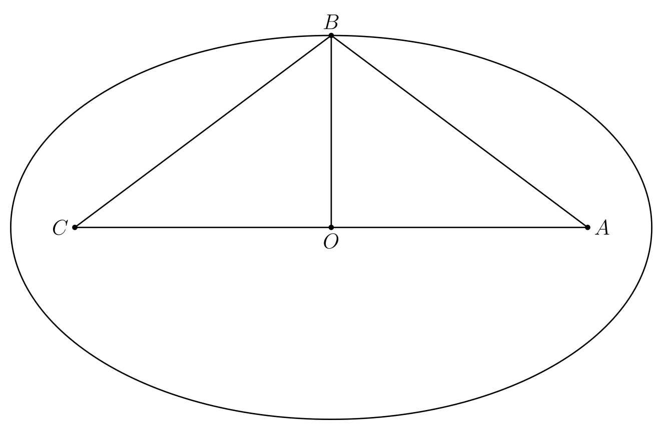

Fig. 6: Configuration with max area

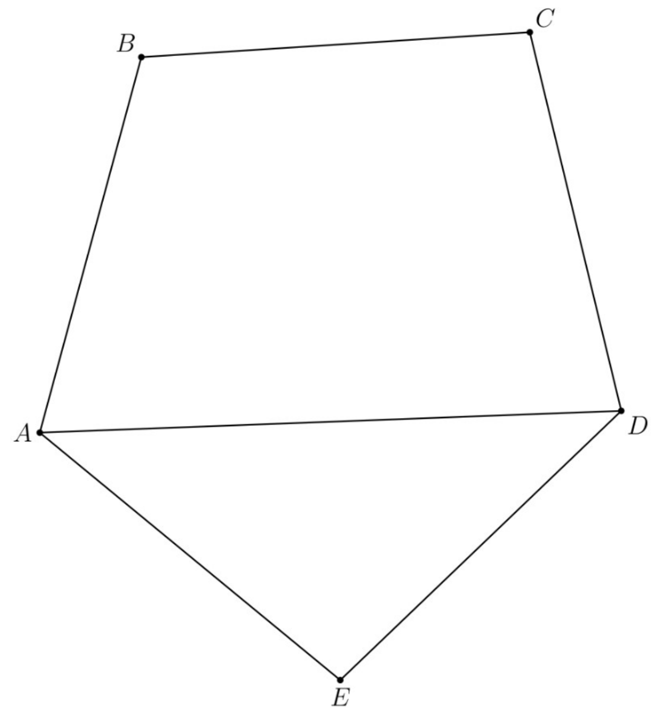

Fig. 7: Example for pentagon, all sides are equal

Now consider three consecutive sides, with vertices A, B, C, and D. We know that $AB = BC = CD$. We claim the area is maximized when the angles $\angle ABC = \angle BCD$.

Consider Bretschneider's Formula, which states that for any quadrilateral, its area is equal to

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)},$$

where s is the semiperimeter, a, b, c, and d are the side lengths, and α and γ are opposite angles. Note that when $ABCD$ is cyclic, the area is maximized, since the term $abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)$ vanishes. Therefore, we want to position the four vertices such that they are cyclic.

Consider $ABCD$ once it is cylic, and let $\angle ABC + \angle ADC = 180°$. Since $AB = BC = CD$, each side cuts out an arc of equal degree. In particular, letting O be the center of the circle, we have that $\angle AOB = \angle BOC = \angle COD$. This means $\triangle AOB \cong \triangle BOC \cong \triangle COD$, so $\angle ABO = \angle OBC = \angle OCB = \angle DCO$.

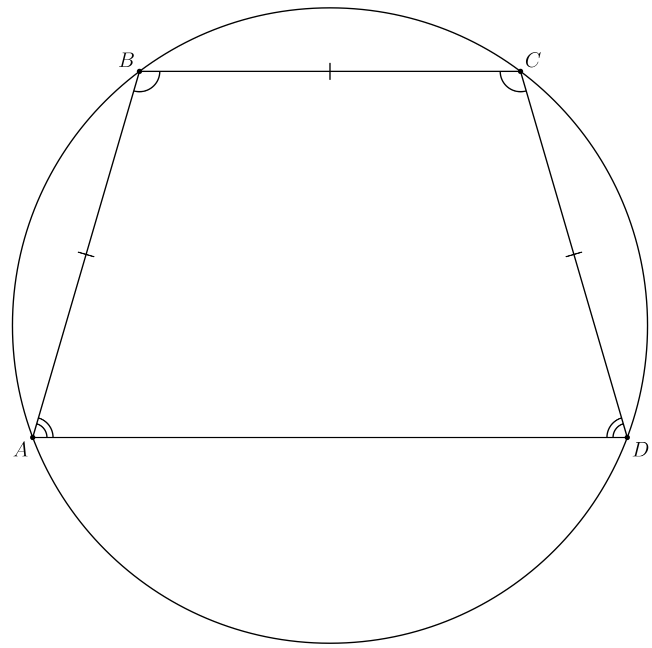

Fig. 8: Configuration with max area

Since $\angle ABC = \angle ABO + \angle OBC$, we have that $\angle ABC = \angle BCD$. Thus, when the area of the quadrilateral is maximized, the angles are equal.

Thus, the area of a polygon S with fixed perimeter p is maximized when S is regular. □

Corollary. *The maximum area of an n-gon with perimeter p is* $\frac{p^2/n}{4\tan(180/n)}$.

Proof: Decompose the regular n-gon (it has the max area) into n congruent isoceles triangles, with the base vertices as two consecutive vertices on the n-gon and the other vertex being the center of the n-gon. The vertex angle is $\frac{360}{n}$, and the other two angles are $90° - \frac{180}{n}$. We know that the side opposite of the vertex is $\frac{p}{n}$. The area of a triangle with all angles and one side is $A = \frac{a^2 \sin B \sin C}{2 \sin A}$, where a is opposite A. Plugging in our values, we obtain that the area of one of these triangles is

$$
\begin{aligned}
A &= \frac{(p/n)^2 \sin^2\left(90^\circ - \frac{180}{n}\right)}{2\sin\left(\frac{360}{n}\right)} \\
&= \frac{(p/n)^2 \cos^2\left(\frac{180}{n}\right)}{4\sin\left(\frac{180}{n}\right)\cos\left(\frac{180}{n}\right)} \\
&= \frac{(p/n)^2}{4\tan(180/n)}.
\end{aligned}
$$

Multiplying this by n for the n triangles gives us the desired area. □

10 Isoperimeteric Problem

We now tackle the original Isoperimetric Problem. We will prove the following inequality:

$$\frac{l^2}{4\pi} \geq A,$$

where l is the length of the curve and A is the area of the curve. Equality occurs when the curve is a circle.

We first prove some lemmas.

Lemma 1. *Given a closed curve C, let $c(t) = (x(t), y(t))$ be its parametrization, with $t \in [a, b]$ and $c(a) = c(b)$. Let the area bounded by the curve be A. Then,*

$$A = \int_a^b xy'\,dt = -\int_a^b x'y\,dt,$$

Proof: By Green's Theorem, we have that

$$A = \oint_C xy' = -\oint_C x'y.$$

Evaluating these gives the desired conclusion.

Lemma 2. *Let x, y, and z be functions of t with continuous first derivatives. We have that*

$$(xy' - zx')^2 \leq (x^2 + z^2)((x')^2 + (y')^2)$$

Proof: By the Trivial Inequality,

$$
\begin{aligned}
0 &\leq (xx' + zy')^2 \\
&= x^2(x')^2 + 2xx'zy' + z^2(y')^2 \\
&= x^2(x')^2 + x^2(y')^2 + z^2(x')^2 + z^2(y')^2 - (x^2(y')^2 - 2xy'zx' + z^2(x')^2) \\
&= (x^2 + z^2)((x')^2 + (y')^2) - (xy' - zx')^2.
\end{aligned}
$$

Equality occurs when $xx' = -zy'$.

We are now ready for the main proof. Let $c(t) = (x(t), y(t))$ be the parametrization of the positively oriented closed curve C with length l. Let $I = [-r, r]$ such that $x(t) \in I$ (graphically, these are two parallel lines tangent to C such that C is entirely between them, as seen below). Without loss of generality, let $x(0) = x(l) = r$ and $x(m) = -r$ for some $0 < m < l$. Define $k(t) = (x(t), z(t))$ to be a circle with radius r ($z(t) = \sqrt{r^2 - x(t)^2}$ for $0 < t \leq m$ and $z(t) = -\sqrt{r^2 - x(t)^2}$ for $m < t \leq l$).

Let A be the area of C and B the area of the circle. By Lemma 1, we have that

$$A = \int_0^l x(t)y'(t)\,dt$$

and

$$B = -\int_0^l x'(t)z(t)\,dt = \pi r^2.$$

Adding these together gives

$$\begin{aligned} A + \pi r^2 = A + B &= \int_0^l xy' - x'z\,dt \\ &\leq \int_0^l \sqrt{(xy' - x'z)^2}\,dt \\ &\leq \int_0^l \sqrt{(x^2 + z^2)((x')^2 + (y')^2)}\,dt \\ &= \int_0^l r\,dt = rl, \end{aligned}$$

where the second inequality comes from Lemma 2, and the second to last equality comes from $x^2 + z^2 = r^2$ and $(x')^2 + (y')^2 = 1$, since the curve is parametrized by arc length. By AM-GM,

$$rl = A + \pi r^2 \geq 2\sqrt{A\pi r^2} \implies r^2 l^2 \geq 4A\pi r^2 \implies \frac{l^2}{4\pi} \geq A,$$

as desired.

To find out when equality occurs, we note that since we used AM-GM at the end, A has to equal πr^2, which means $l = 2\pi r$. By Lemma 2, equality between the second and third integral occurs when $-xx' = zy'$. Note that $x^2 + z^2 = r^2$ and $(x')^2 + (y')^2 = 1$. From here, we reduce the equality $(xy' - zx')^2 = (x^2 + z^2)((x')^2 + (y')^2)$ into a function of y'.

$$(xy' - zx')^2 = (x^2 + z^2)((x')^2 + (y')^2) \implies x^2(y')^2 - 2zx'xy' + z^2(x')^2 = r^2$$

$$\implies x^2(y')^2+2(x')^2x^2+z^2(x')^2 = r^2 \implies x^2(y')^2+(x')^2x^2+(x')^2x^2+z^2(x')^2 = r^2$$

$$\implies x^2((x')^2+(y')^2)+(x')^2(x^2+z^2) = r^2 \implies x^2+(x')^2(x^2+(r^2-x^2))$$

$$\implies x^2+(x')^2r^2 = r^2 \implies x^2 = r^2(1-(x')^2) \implies x^2 = r^2(y')^2 \implies x = \pm ry'.$$

Now we show that $y = \pm rx'$. We now basically repeat the entire above argument from proving the inequality to achieving this equality case, except when we parametrize C, we now bound it by two parallel lines except these parallel lines will be perpendicular to the intial two parallel lines. The lines will bound the curve in an interval $I = [-r', r']$, and the curve will be parametrized as $c'(t) = (w(t), y(t))$, where $y(t)$ is the same as above. Repeating the procedure we get that $A = \pi(r')^2$, but A has the same area in both cases, so $r' = r$. Similarly, $-xw' = yy'$, so after the equality calculation, we get that $y \pm rx'$ (this whole entire paragraph is basically saying that we are switching the x and y-axes in our initial parametrization).

Finally, squaring both equalities with x and y and adding them gives

$$x^2+y^2 = r^2((x')^2+(y')^2) = r^2,$$

which is a circle, so we are done. □

Interestingly enough, isoperimetric problems can be generalized to higher dimensions, where instead of perimeter and area, $n-1$-dimensional measure and n-dimensional measure are used. This topic is beyond the scope of this paper, so it won't be covered, but more info can be found here [2].

References

[1] Jian Liu. Further generalization of walker's inequality in acute triangles and its applications. *Aims Mathematics*, 5(6):6657–6672, 2020. 1

[2] Li Xuanyu. Isoperimetric inequality: Its origin, proof and development. 2022. 1