Integral Geometry Overview

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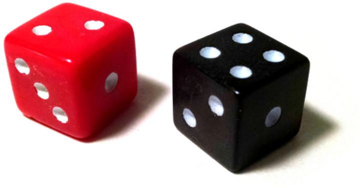
In this talk, I will go over the following topics:

Introduction

Definition: Integral geometry, which can also be called geometric probability, is the study of measures on a geometric space, invariant under a symmetry group. From integral geometry arose various fields of study including geometric measure theory, stereometry, and tomography.

An Example



As an example, one of the problems integral geometry can solve is the following: if two cubes of equal size collide, what is the probability that the collision is corner-to-face? The answer is roughly 0.46.

Unoriented Lines

An unoriented line L in the plane is given by the following two quantities: $p \geq 0$ the closest distance from a point on the line to the origin (which is called the foot point) and $0 \leq \theta < 2\pi$, the angle formed between the segment containing the foot point and the origin, and the positive $x - axis$. From here the equation of the line follows:

$$\cos\theta x + \sin\theta y = p.$$

Rigid Motion

A rigid motion $\mathcal{M}$ of a set of points is defined as a rotation by an angle of ϕ and then a translation by a vector (x_0, y_0). From here, it is clear that a rigid motions can map any unoriented line onto another. The formula for rigid motion is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Kinematic Measure

To do the probability calculations we want in integral geometry, we want to be able to integrate with respect to a measure for a set of lines that is invariant under rigid motions. This measure is called kinematic measure, and is given by

$$dK = dp \wedge d\theta$$

The proof of this involves a simple calculation with the Jacobian.

Poincare's Formula for Lines

Poincare's Formula for Lines gives the measure of unoriented lines meeting a curve in the plane.
If C is a piecewise $\mathcal{C}^1$ curve in the plane, then the measure of unoriented lines meeting C (with multiplicity) is given by

$$2\text{L}(C) = \int_{L:L\cap C\neq\emptyset} n(C\cap L)dK(L)$$

Outline of Proof of Poincare's Formula for Lines

To prove Poincare's Formula for Lines, we consider the set S defined by

$$S = \{(L, Z); L \cap C \neq \emptyset, Z \in L \cap C\}.$$

The unoriented lines can be measured first by just considering lines meeting C, and then considering the points on C and lines passing through them.

Theorem (Sylvester's Problem)

If Ω is a convex bounded region, we can expected almost all unoriented lines that meet it to meet it twice. The same holds for $\omega \subset \Omega$. Therefore, using conditional probability, we find the following result:

$$P = \frac{\mathsf{L}(\partial\omega)}{\mathsf{L}(\partial\Omega)}$$

Image Credit: Neil Kolekar

Buffon's Needle and Pi

This and another formula which is in my paper can be used to solve Buffon's Needle problem, which asks if a needle of length $l \leq d$ is dropped on a plane with infinite parallel lines spaced d apart, what is the probability that the needle meets one of the lines. The answer turns out to be $\frac{2l}{\pi d}$. From this, an experimental determination of π was done by dropping 5000 needles in 1850 by Wolf, and the value found was 3.1596.

Thank You

I would like to thank the TAs, my TA Alphonse, Simon, as well as all of my classmates for helping me learn in this Euler Circle Class. To conclude,

I will end with a picture of a duck.