INTEGRAL GEOMETRY EXPOSITORY PAPER

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1. Abstract

This paper will go into detail in explaining unoriented lines, kinematic measure, Poincare's Formula for Lines, as well as other fundamentals in integral geometry. Interesting examples will also be provided as well as intuitive insight.

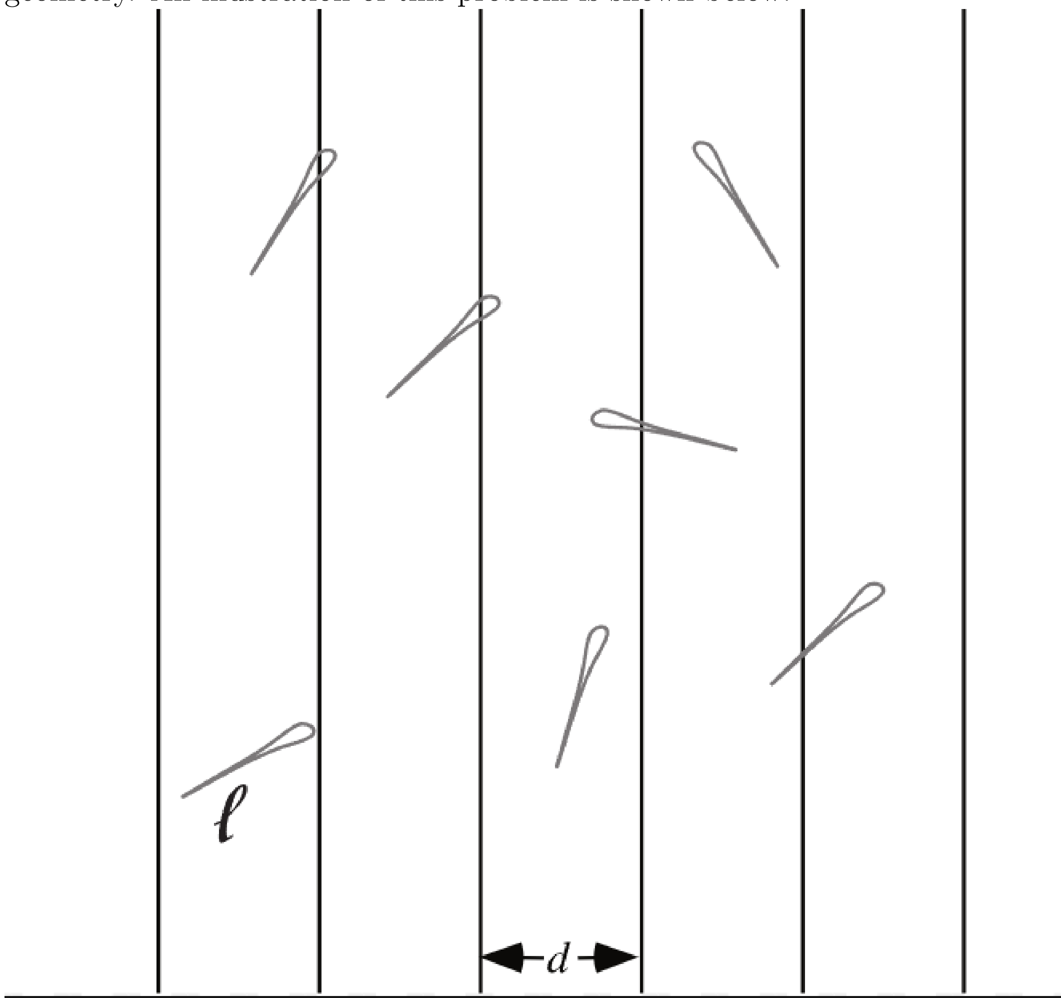
Contents

2. Introduction

Integral geometry is defined as the study of measures on geometric spaces that are invariant on a symmetry group. Integral geometry has always fascinated me because it allows us to carry out calculations that seem super interesting, like on the way objects collide or the expected number of intersections of lines within a convex region, but seem super difficult to approach in other ways. One of the most famous problems in Integral Geometry is the Buffon's Needle Problem, which involves the following conditions:

(1) There is a plane, with infinite parallel lines on it, each a distance d apart
(2) There is a needle, of length $l \leq d$

The question is, if the needle is randomly dropped on the plane, what is the probability that it touches one of the lines? To solve this problem, we need to use the techniques of integral geometry. An illustration of this problem is shown below.



Caption: An illustration of Buffon's Needle. Credit: Wolfram MathWorld

Another application of integral geometry can be used to solve the following problem involving two cubes of equal size: if two cubes of equal size randomly collide, what is the probability that the collision is corner to face?

Integral can give us the answer to this problem: the probability of a corner to face collision is approximately 0.46.

3. Brief History

Integral geometry was first worked on by Luis A. Santalo and Wilhelm Blaschke. From integral geometry came many other fields of study, including geometric measure theory, stereometry, and tomography. These fields of study can have interesting applications in even x-rays.

4. Kinematic Measure

In integral geometry, it is often useful to be able to accurately describe all lines in a way that they can be measured equally. Therefore, we use unoriented lines, defined as follows.

Definition 4.1. An unoriented line L in the plane is given by two variables, $p \geq 0$ and $0 \leq \theta < 2\pi$, where p is the closest distance from a point on L to the origin and θ is the angle formed between the segment connecting the closest point and the positive x-axis (or, equivalently, the ranges $-\infty < p < \infty$ and $0 \leq \eta < \pi$ may be used). This is shown in the following diagram:

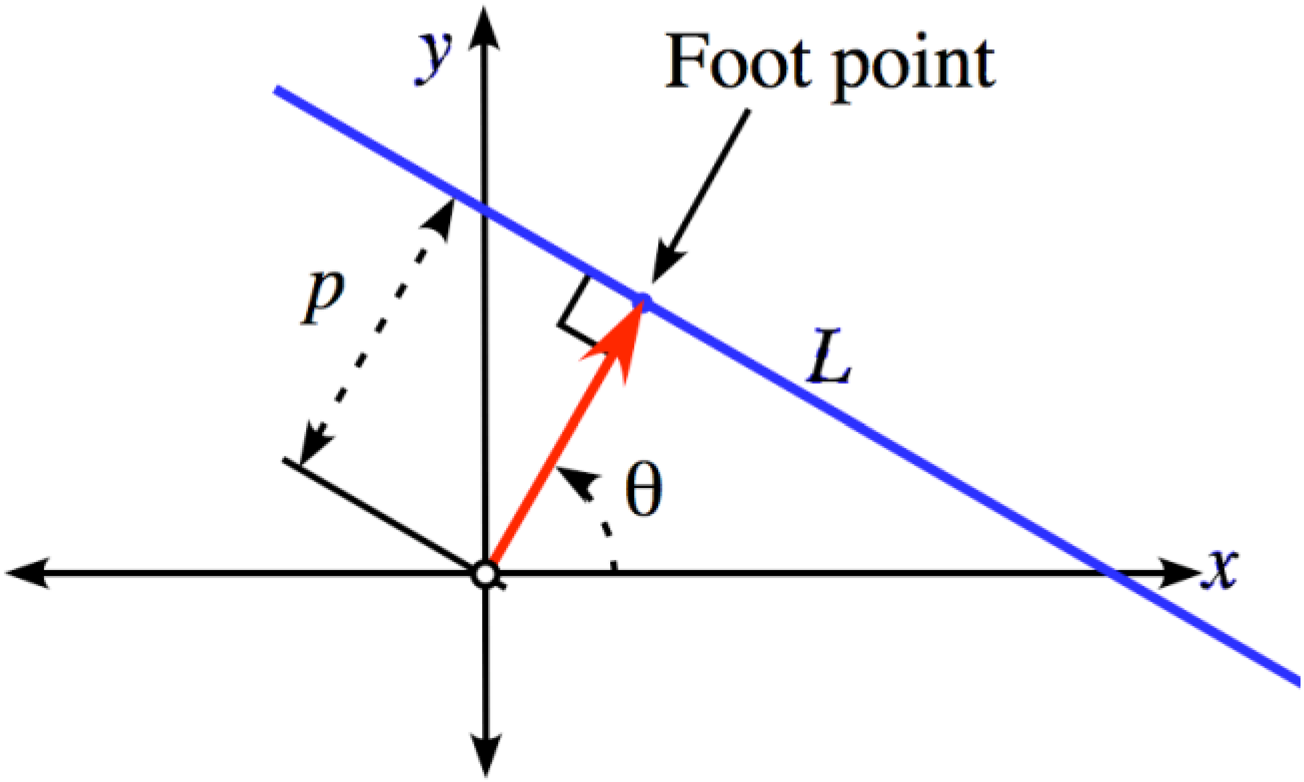


Image Credit: http://www.math.utah.edu/ treiberg/IntGeomSlides.pdf

Consider the point P on the line L which is closest to the origin. It is clear that the coordinates of P must be $(p\cos\theta, p\sin\theta)$. Furthermore, the segment from the origin to P has slope

$$m = \frac{p\sin\theta}{p\cos\theta} = \frac{\sin\theta}{\cos\theta}.$$

This segment is perpendicular to the line L, and it follows that the slope of the line L must be

$$\frac{-\cos\theta}{\sin\theta}.$$

Therefore, the equation of line L is

$$\cos(\theta)x + \sin(\theta)y = (p\cos\theta\cos\theta) + (p\sin\theta\sin\theta)$$
$$\cos(\theta)x + \sin(\theta)y = p(\cos^2\theta + \sin^2\theta)$$
$$\cos(\theta)x + \sin(\theta)y = p(1)$$
$$\cos(\theta)x + \sin(\theta)y = p.$$

Note that this allows us to, given a point on an unoriented line as well as the angle of that line, find the value of p easily.

Proposition 4.2. *A rigid motion M on an unoriented line is a translation by (x_0, y_0) followed by a rotation by an angle ϕ. Specifically, we have the following:*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Therefore, the following describes the inverse of rigid motion, M^{-1}

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

To do integral geometry, it is necessary to find a measure on unoriented lines that is invariant under rigid motion. This measure will be called *kinematic measure*.

Definition 4.3. For an unoriented line in (p, θ) coordinates, the kinematic measure is given by

$$dK = dp \wedge d\theta.$$

We will now prove the following theorem:

Theorem 4.4. *(Kinematic Measure of Unoriented Lines is invariant under rigid motion) Suppose L is an unoriented line. Then we have the following:*

$$dK(L) = dK(\mathcal{M}(L))$$

where $\mathcal{M}$ is a rigid motion.

Proof. Let's say that the rigid motion consists of first a rotation about the origin by an angle ϕ as well as a translation by (x_0, y_0). Following the rotation, our line given by (p, θ) coordinates is just $(p, \theta + \phi)$ as the distance from the line to the origin is preserved. After the translation, the angle is preserved, and we have

$$p' = p + \cos(\theta + \phi)x_0 + \sin(\theta + \phi)y_0.$$

Hence, the new line $L' = M(L)$ is determined by

$$(p', \theta') = (p + \cos(\theta + \phi)x_0 + \sin(\theta + \phi)y_0, \theta + \phi).$$

The Jacobian Formula for change in measure gives the following

$$dK(L') = |J| dK(L).$$

where

$$J = \frac{\partial(p', \theta')}{\partial(p, \theta)} = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial p'}{\partial \theta} \\ \frac{\partial \theta'}{\partial p} & \frac{\partial \theta'}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix} = 1.$$

Hence, $dK(L') = dK(L)$, implying the kinematic measure is invariant under rigid motion, as desired. ■

Alternatively, this can be proven using properties of the wedge product. The wedge product is a skew product, implying that it satisfies the following properties:

Proposition 4.5. *(Properties of a Skew Product *)*

1. $a * a = 0$ *for all* a.
2. *Distributes over addition.*

From here we can prove the following about the skew product:

Lemma 4.6. *For a skew product* $*$ *and for all* a, b*, we have* $(a * b) = -(b * a)$.

Proof. We have the following

$$\begin{aligned} &0 \\ &= (a + b) * (a + b) \\ &= (a * a) + (a * b) + (b * a) + (b * b) \\ &= 0 + (a * b) + (b * a) + 0 \\ &= (a * b) + (b * a). \end{aligned}$$

■

From here it follows that

$$(a * b) = -(b * a).$$

Next, we note that

$$\begin{aligned} &p' = p + \cos(\theta + \phi)x_0 + \sin(\theta + \phi)y_0 \\ &\theta' = \theta + \phi. \end{aligned}$$

From here it follows

$$\begin{aligned} &dp' = dp + (-\sin(\theta + \phi)x_0) + (\cos(\theta + \phi)y_0). \\ &d\theta' = d\theta. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&dp' \wedge d\theta' \\
&= (dp + (-\sin(\theta+\phi)x_0) + (\cos(\theta+\phi)y_0 d\theta) \wedge d\theta \\
&= dp \wedge d\theta + ((-\sin(\theta+\phi)x_0) + (\cos(\theta+\phi)))(d\theta \wedge d\theta) \\
&= dp \wedge d\theta + 0 \\
&= dp \wedge d\theta
\end{aligned}$$

This completes the alternative proof of the invariance of kinematic measure. ■

5. Proof of Poincare's Formula for Lines 1896

Let C be a piecewise C^1 curve in the plane.

Definition 5.1. For an unoriented line L in the plane, $n(L \cap C$ is the number of intersectoin points between the line L and the curve C. If C contains a linear segment that agrees with L, then $n(L \cap C)$ is taken to be ∞.

In the definition it does say that $n(L \cap C)$ can have a value of ∞, but this will not matter when considering the the measure of unoriented lines meeting a curve, in Poincare's Formula. Intuitively, if a curve happens to have a straight line segment, then there will be an unoriented line agreeing with that curve, but in the scope of all unoriented lines, this will not matter when integrating, as the dK-measure of such lines will be 0. Now, we state Poincare's Formula for Lines.

Theorem 5.2. *Let C be a piecewise $\mathcal{C}^1$ curve. Then, the measure of unoriented lines meeting C, counting with multiplicity, is given by*

$$2\mathrm{L}(C) = \int_{\{L:L\cap C\neq\emptyset\}} n(C \cup L)dK(L).$$

Proof. We will first solve the problem for C^1 curves, then we can add the integrals together to get the any curve C.

Assume C is a C^1 curve. We parameterize C by arclength as $Z(s) = (x(s), y(s))$. Therefore there are $x(s), y(s) \in C^1[0, s_0]$ such that the tangent vector

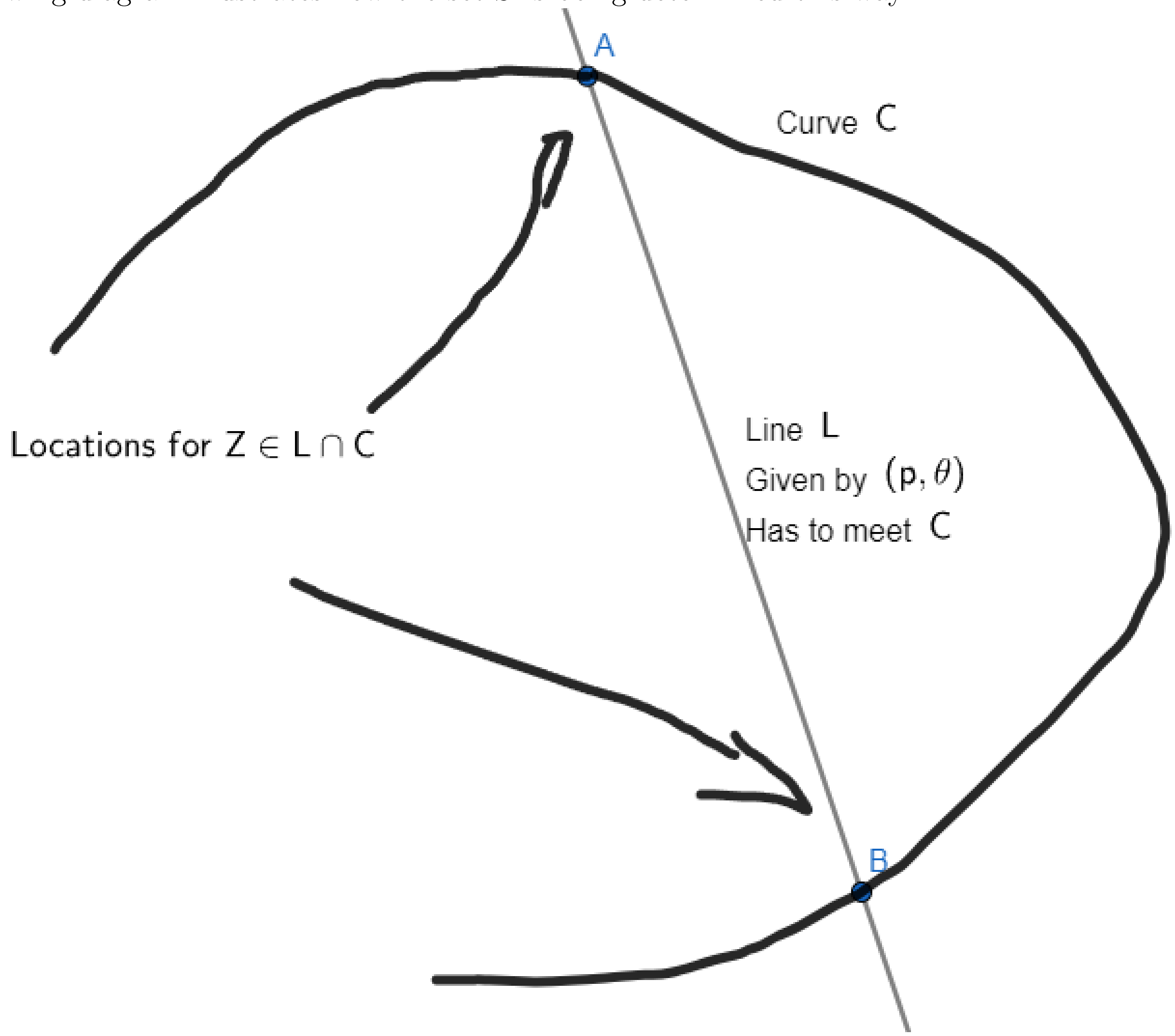
$$\dot{Z} = (\dot{x}, \dot{y})$$

satisfies $|\dot{Z}| = 1$ (because we have parameterized by the arc length).

In order to prove Poincare's Formula for Lines, we will consider a flag that is the set of pairs (L, Z) where L is a line in the plane and $Z \in L$ is a point on the line L. The set of points and lines that we are interested in is the following subset of the flag

$$S = \{(L, Z); L \cap C \neq \emptyset, Z \in L \cap C\}.$$

The line L is determined by the coordinates (p, θ). The point $Z \in L$ is determined by an arc length coordinate q along L from the foot point of the line L which is $(p\cos\theta, p\sin\theta)$. The following diagram illustrates how the set S is being determined this way.



Caption: In this diagram, the curve C is shown and one of the lines L meeting C at at least one point is also shown. Futhermore, the possible locations $Z \in L \cap C$ are shown.

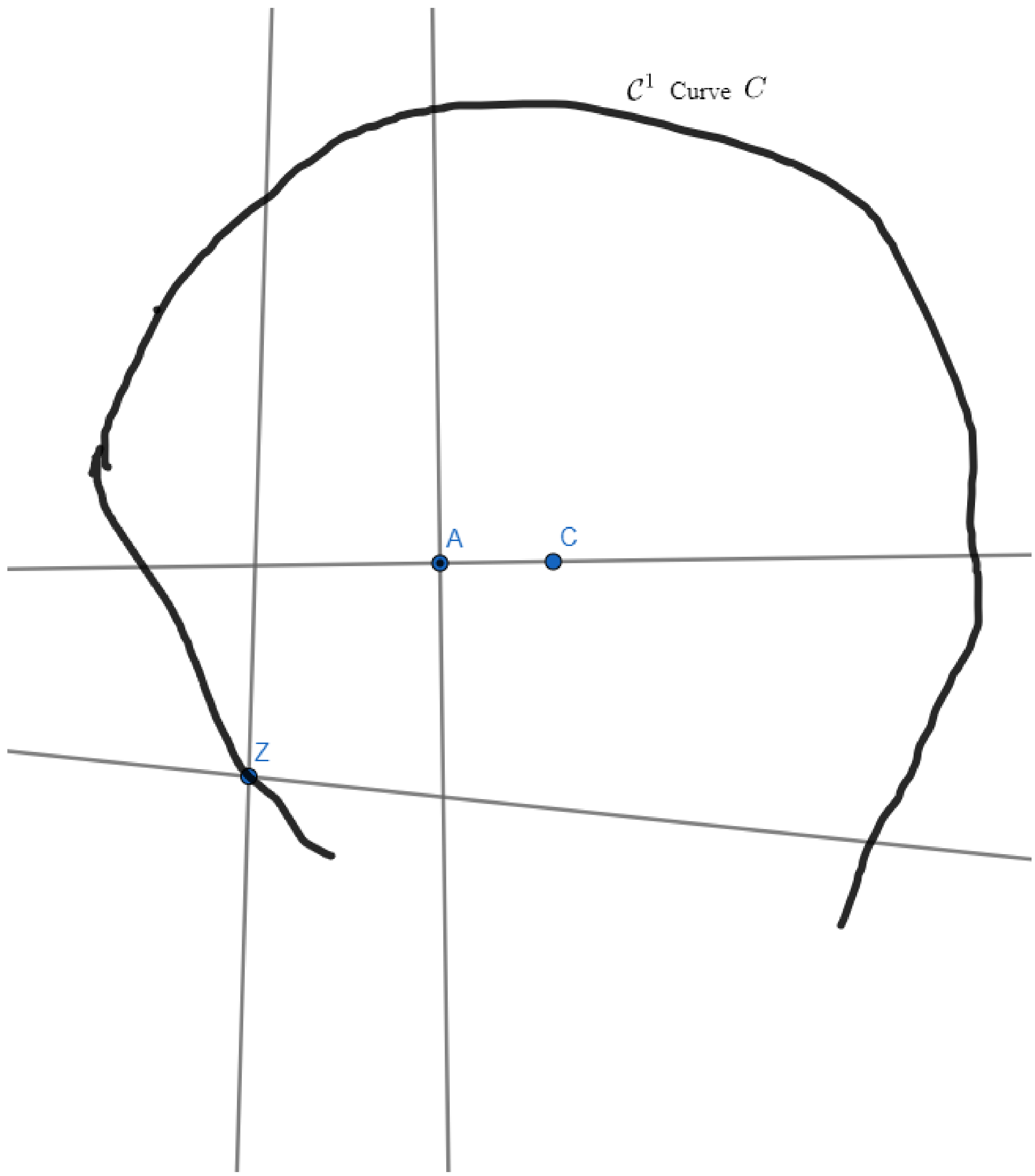
In this way of describing S, we are first picking a line that intersects with C and then looking at the intersection points. From here, we get the following:

$$\int_{L:L\cap C\neq\emptyset} n(C\cap L)dK(L) = \int_{L:L\cap C\neq\emptyset} \left(\sum_{Z\in L\cap C} 1\right) dK(L)$$

To prove Poincare's Formula, we will look at another way to determine S. This is by first selecting the point $(x, y) = Z \in C$ and then taking L to be any unoriented line such that $Z \in L$ having an angle $0 \le \eta < \pi$ (we are only going up till π because otherwise we will get two orientiations of the same line passing through Z. We have

$$\tilde{p} = x(s)\cos\eta + y(s)\sin\eta.$$

Again, we can alternatively use $-\infty < \tilde{p} < \infty$ and $0 \leq \eta < \pi$ instead of $p \geq 0$ and $0 \leq \theta < 2\pi$. The motivation for choosing this new range is that it is unclear if the value of $\tilde{p}$ will be positive or negative for the line L we choose passing through Z. The following diagram illustrates this concept, as well as the concept of picking unoriented lines through the same point Z.



Caption: In this diagram, we can clearly see, for the point Z which is on the curve C, the value of p will be negative. Furthermore, this diagram illustrates the other method of looking at unoriented lines, by first selecting a point on the curve then looking at all the unoriented lines that pass through it.

It might seem as though by picking unoriented lines through a point on the curve C we might pick the same line multiple times. However, it is important to note that the number of times we pick the same line L is just $n(L \cap C)$, the number of intersection points, which is exactly what we want.

We proceed, taking the differential of $\tilde{p}$ as follows

$$d\tilde{p} = (\dot{x}(s)\cos\eta + \dot{y}(s)\sin\eta)ds + (-x(s)\sin\eta + y(s)\cos\eta)d\eta$$

Recall that we have chosen $x(s), y(s)$ such that the magnitude of the tangent vector is 1. Therefore, we can describe $(\dot{x}(s), \dot{y}(s))$ as $(\cos\phi(s), \sin\phi(s))$.
From here it follows that

$$\begin{aligned} & d\tilde{p}d\eta \\ &= |\begin{vmatrix} \cos\phi\cos\eta + \sin\phi\sin\eta & * \\ 0 & 1 \end{vmatrix}| ds d\eta \\ &= |\cos(\phi(s) - \eta)| ds d\eta \end{aligned}$$

Then, we can evaluate the following integral

$$\begin{aligned} & \int_{L: L \cap C \neq} \left(\sum_{Z \in L} 1 \right) dK \\ &= \int_{Z: Z \in C} \int_{L: Z \in L} d\tilde{p} d\eta \\ &= \int_0^{s_0} \int_0^{\pi} |\cos(\phi - \eta)| d\eta ds \\ &= 2 \int_C ds \\ &= 2\text{L}(C) \end{aligned}$$

To evaluate the integral of $|\cos(\phi - \eta)|$ we can make the observation that it will always be 2 no matter what ϕ is because it will always come out to equal the area of the positive section of the cosine function in one period. This completes the proof of Poincare's Formula. ■

6. Sylvester's Problem

Theorem 6.1. *(Sylvester's Problem) If Ω is a bounded convex region and ω is another bounded convex region such that $\omega \subset \Omega$, then let P be the probability that a line meets ω given that it meets Ω. P is given by*

$$P = \frac{L(\omega)}{L(\Omega)}$$

Proof. Since ω and Ω are convex regions, almost all lines that meet them will intersect the boundary twice. Furthermore, if any line meets ω, it must also meet Ω, as ω is a subset. Therefore, we can apply Poincare's Formula for Lines to get the following

$$\begin{aligned} P &= \frac{2\mathrm{L}(\omega)}{2\mathrm{L}(\Omega)} \\ &= \frac{\mathrm{L}(\omega)}{\mathrm{L}(\Omega)} \end{aligned}$$

■

7. Acknowledgements

I would like to thank Simon, my TA Alphonse, all of the TAs at Euler Circle and my classmates for helping me with my research. I would also like to thank them for providing resources and creating an environment where I can pursue my interest in math.

References

[1] Andrejs Treibergs. Integral geometry and geometric probability. 1, 2008.
[2] Luis A. Santalo. *Integral Geometry and Geometric Probability*. Addison-Wesley, 1976.

[1] [2]