Lattice Path Enumeration

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Introduction

Motivation

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- The **step set** of a lattice path model is a fixed set containing all possible steps in the paths of interest.
- We present an example: Dyck paths.

Motivation (Example: Dyck paths)

- **Dyck paths**: paths from $(-n, 0)$ to $(n, 0)$ that do not go below the x-axis where the steps are in $\{\langle 1, 1\rangle, \langle 1, -1\rangle\}$.

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- Dyck paths are enumerated by the Catalan numbers, given by $C_n := \frac{1}{n+1}\binom{2n}{n}$.
- There are numerous ways of deriving the enumeration of Dyck paths. One common method is to recursively compute C_n by strong induction on n, whence we approach the recursion

$$C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}.$$

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- In most enumerative combinatorics problems, finding a generating function that enumerates a combinatorial object of interest is useful.
- The reason why generating functions are so useful is because they can be **manipulated in various ways** that combinatorial quantities cannot.

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As an example, we consider Dyck paths. Using only elementary methods (such as bijections) we find the recurrence

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$$C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}.$$

Let the generating function for Dyck paths by $C(x)$. From power series multiplication we quickly find that

$$C(x) = xC(x)^2 + 1,$$

from which we can solve for $C(x)$ as $C(x) = \frac{1-[1-4x]^{1/2}}{2x}$.

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- Here "information" can be things like:
  - explicit forms of sequences for enumeration;
  - explicit forms of their corresponding generating functions;
  - determination of algebraicity/transcendence and D-finiteness of generating functions, etc.

Prerequisites

We briefly go over some general terminology that will be used extensively in the remainder of this talk.

- *Analytic prerequisites*
- *Generating functions as combinatorial objects*

Prerequisites – Analytic Terminology

Definition (Generating Functions)

The **generating function** of a d-dimensional sequence $\{a_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{Z}^d_{\geq 0}}$ is the d-variate power series

$$F(x_1, \ldots, x_d) = \sum_{\langle j_1,\ldots,j_d\rangle \in \mathbb{Z}^d_{\geq 0}} a_{\langle j_1,\ldots,j_d\rangle} x_1^{j_1} \cdots x_d^{j_d}.$$

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Definition (Diagonals)

Consider the generating function F from the above definition where $d = 2$. Then the diagonal $\triangle F$ of F is

$$\triangle F(x) := \sum_{i\geq 0} a_{\langle i,i\rangle} x^i.$$

Prerequisites – Analytic Terminology

Definition (Algebraicity)

We say that a Laurent series $F(X) = \{a_n\}_{n \geq -n_0}$ is **algebraic** over the field of characteristic 0 ($\mathbb{K}[X]$) if for some integer d there exists polynomials $A_0(X), \ldots, A_d(X)$ with coefficients in K and not all 0 such that

$$\sum_{i=0}^{d} A_i(X)F(X)^i = 0.$$

Laurent series that are not algebraic are called transcendental.

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Prerequisites – Analytic Terminology

Definition (D-finite functions)

A function $f = f(x)$ is called **D-finite** (or holonomic) if there exist polynomials $0 \neq a_0(x), \ldots, a_r(x) \in \mathbb{K}[X]$ such that

$$\sum_{k=0}^{r} a_k(x) f^{(k)}(x) = 0.$$

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A useful fact on the D-finiteness of bivariate power series is the following.

Theorem

If $f(x_1, x_2)$ is a D-finite generating function, then so is its diagonal $\triangle f(x)$.

Prerequisites – Analytic Terminology

Theorem (Lagrange inversion)

Let $G(t)$ be any element of $\mathbb{C}[[t]]$. Then the equation $f(y) = yG[f(y)]$ has a unique solution in $\mathbb{C}[[y]]$, and

$$\langle y^n \rangle (f(y))^n = \frac{k}{n} \langle t^{n-k} \rangle (G(t))^n \qquad \forall n, k > 0 \qquad (1)$$

$$\langle y^n \rangle \frac{(f(y))^k}{1 - yG'(f(y))} = \langle t^{n-k} \rangle (G(t))^n \qquad \forall n, k \geq 0 \qquad (2)$$

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- In general, operations such as multiplication, differentiation, and diagonalization have combinatorial significance.

Prerequisites – Generating functions in combinatorics

Example (Distributions)

Show that the number of solutions in nonnegative integers to $a_1 + \ldots + a_k = n$ is $\binom{n+k-1}{k-1}$.

Proof.

- The generating function for the number of solutions is $f(x) := (1 + x + x^2 + \ldots)^k = (1 - x)^{-k}$.
- Upon routine power series manipulation, it is easy to see that the coefficient of x^n is f is indeed $\binom{n+k-1}{k-1}$.

□

Kernel Methods

Introduction to Kernel Methods

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- For a lattice path model with step set $\mathcal{S}$, we say its **characteristic polynomial** is the Laurent polynomial

$$S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} w_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}.$$

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  By default, the weights $w_{\mathbf{i}}$ are set to 1.
- We define a **kernel** $K(t, \mathbf{z})$ for each model. The definitions may vary, but they are generally similar, in that they are a function of t and $S(\mathbf{z})$.
- In several general lattice path models, we can derive several **kernel equations** that contain kernels. This gives us a way of computing generating functions that count certain lattice paths.

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- In this case, the kernel is $K(x,t) = 1 - txS(x)$.
- Let
$$W(\mathbf{z},t) := \sum_{n\geq 0}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} f_{\mathbf{i},n}\mathbf{z}^{\mathbf{i}}\right)t^n,$$
where $f_{\mathbf{i},n}$ represents the number of walks in the lattice model of interest which consists of n steps and end at a lattice point $\mathbf{i}\in\mathbb{Z}^d$.

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where $f_{\mathbf{i},n}$ represents the number of walks in the lattice model of interest which consists of n steps and end at a lattice point $\mathbf{i}\in\mathbb{Z}^d$.
- The kernel equation in this case is
$$W(\mathbf{z},t)\cdot K(\mathbf{z},t) = \mathbf{z}^{\mathbf{p}}.$$

Unrestricted models

Example

The number of walks that both start and end at the origin has its generating function given by

$$\begin{aligned} E(t) &= [\mathbf{z}^0] W(\mathbf{z}, t) \\ &= \triangle W(\mathbf{z}, z_1 z_2 \cdots z_d t) \\ &= \triangle \left(\frac{\mathbf{z}^{\mathbf{p}}}{1 - t(z_1 \cdots z_d) \cdot S(\mathbf{z})} \right), \end{aligned}$$

which is D-finite since diagonals preserve D-finiteness.

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- We say that the **small roots** of a equation solving for power series in t are the roots that approach 0 as t approaches 0.
- Here we consider kernel $K(x, t) = 1 - txS(x)$.

One-dimensional excursions

Theorem (kernel equation for one-dimensional excursions)

Let $\mathcal{S} \subset \mathbb{Z}$ be a step set with smallest element $-m$ where $m > 0$. The generating function $E(t)$ that enumerates walks with steps in $\mathcal{S}$ which begin and end at the origin is given by

$$E(t) = t \sum_{i=1}^{m} \frac{r_j'(t)}{r_j(t)},$$

where $r_1(t), \ldots, r_m(t)$ are the small roots of $K(x,t) := 1 - tS(x)$ in x.

Walks in Half-space

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- This induces a new **weighted** step set $\mathcal{S}$.
- Let $H(x,t) \in \mathbb{R}[x][[t]]$ be the power series defined by

$$H(x,t) = \sum_{n \geq 0} \sum_{i \geq 0} h_{i,n} x^i t^n,$$

  where $h_{i,n}$ enumerates the weighted half-space walks of length n on the steps in $\mathcal{S}$ ending at a point with x-coordinate i.

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- We can now deduce a kernel equation for this model; in this case, the kernel is $K(x, t) = 1 - tS(x)$.

Walks in Half-space

Theorem (kernel equation for walks in half-space)

We have

$$K(x,t)H(x,t) = 1 - t\sum_{j=0}^{m-1} S_{<-j}(x)x^j\langle x^j\rangle H(x,t).$$

Walks in Half-space

Example (Dyck paths and prefixes in half-space)

Let $\mathcal{S} = \{\bar{1}, 1\}$ be the unweighted step set with characteristic polynomial $S(x) = x^{-1} + x$. By the kernel equation for walks in half-space,

$$\begin{aligned}(1 - t(x^{-1} + x))H(x,t) &= 1 - tx^{-1}\langle x^0 \rangle H(x,t) \\ &= 1 - tx^{-1}H(0,t).\end{aligned}$$

We can find from the above that

$$H(0,t) = \frac{1 - [1 - 4t^2]^{1/2}}{2t^2} = \sum_{k=0}^{\infty} C_k t^{2k}.$$

Conceptually, the number of walks of length $2n$ with step set $\mathcal{S}$ is C_n.

Walks in the Quarter-plane

- We consider walks on a weighted step set $\mathcal{S}' \subset \mathbb{Z}^d$ restricted to the **quarter-space** $\mathbb{Z}^{d-2} \times \mathbb{N}^2$.
- Rather than taking an arbitrary unweighted step set $\mathcal{S}$, we restrict our model to a **short step** model. This effectively sets $\mathcal{S} \subset \{-1, 1, 0\}^d$.
- The main difference between this new model and the other discussed models is that our kernel becomes a quadratic in its three variables.

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Theorem

The kernel equation for walks in the quarter-plane is given by

$$\begin{aligned} & xy(1 - tS(x,y))Q(x,y,t) \\ = xy - tI(y) - tJ(x) + & \mathbf{1}_{(-1,-1)\in\mathcal{S}} tQ(0,0,t), \end{aligned}$$

where $I(y) := y(\langle x^{-1}\rangle S(x,y))Q(0,y,t)$ *and* $J(x) := x(\langle y^{-1}\rangle S(x,y))Q(x,0,t)$.

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where $I(y) := y(\langle x^{-1}\rangle S(x, y))Q(0, y, t)$ *and*
$J(x) := x(\langle y^{-1}\rangle S(x, y))Q(x, 0, t)$.

- The additional variable in the kernel for quadrant walks is a new conflict because we need to consider algebraic surfaces defined by $K(x, y, t) = 0$. To avoid this conflict, we introduce the **algebraic kernel method**, where we don't have to solve for the kernel.

Group Models

Introduction

- We use group models to use the algebraic kernel method. In this we introduce a group $\mathcal{G}$ consisting of substitutions that fix K.
- Let $\mathcal{G}$ be generated by the involutions Ψ and Φ under composition.
- For example, we can consider step set $\{(0, \pm 1), (\pm 1, 0)\}$, where $\mathcal{G}$ is generated by $\Psi(x, y) = (x^{-1}, y)$ and $\Phi(x, y) = (x, y^{-1})$.

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The kernel equation for finite group models is

$$\sum_{g \in \mathcal{G}} sgn(g) g(xyQ(x,y,t)) = \frac{1}{1 - tS(x,y)} \sum_{g \in \mathcal{G}} sgn(g) g(xy).$$

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Theorem (a condition for finite groups)

Let $\mathcal{S}$ be a 2-dimensional short step model not contained in the half-plane. Then:

1. *There is a unique vanishing point of $S(x,y)$ with positive coordinates (a,b).*
2. *If $\mathcal{G}$ is finite then $\frac{S_{xy}(a,b)}{\sqrt{S_{xx}(a,b)S_{yy}(a,b)}} = \cos(\theta)$ for some rational multiple θ of π.*

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Theorem (non-D-finiteness criterion)

Let S be a two-dimensional small step model not contained in a half-plane. As long as $\frac{S_{xy}(a,b)}{\sqrt{S_{xx}(a,b)S_{yy}(a,b)}}$ cannot be written as $\cos(\theta)$ for some rational multiple θ of π, the generating function $Q(0,0,t)$ in t is non-D-finite.

Laurent Series Factorization

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In 1978, Gessel showed that each Laurent series $f \in \mathbb{C}[[y, y/t]]$ with constant term 1 has a unique **decomposition** $f(t) = f_- f_0 f_+$, where f_- contains only negative powers of t (and constant term 1), f_+ contains only positive powers of t (and constant term 1), and f_0 is independent of t (and constant term 1).

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Lemma (Gessel)

Let $h(t, y) = \sum_{i=0}^{\infty} a_i t^i$, where $a_0, a_1 \in y\mathbb{C}[[y]]$ and $a_i \in \mathbb{C}[[y]]$. Then the equation $\alpha = h(\alpha, y)$ has a unique solution in $\alpha \in y\mathbb{C}[[y]]$. Define $f := [1 - t^{-1}h(t, y)]^{-1}$. Then in the decomposition of $f = f_- f_0 f_+$, we have $f_- = [1 - t^{-1}\alpha]^{-1}$ and $f_0 = \frac{\alpha}{a_0}$. Furthermore, for $k > 0$,

$$\alpha^k = \sum_{n=1}^{\infty} \frac{k}{n} \langle t^{n-k} \rangle [h(t, y)]^n.$$

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- We can also take products of paths; given two paths π_1 and π_2, their product $\pi_1\pi_2$ is the path π_1 followed by π_2.
- A **head** of a path π is any π_1 such that there exists π_2 where $\pi_1\pi_2 = \pi$.

Terminology

Definition (enumeration function)

For a set P of lattice paths over $\mathbb{Z}^2$, we define its **enumeration function** by

$$\vartheta(P)(t, y) = \sum_{\pi \in P} t^{e_x(\pi) - e_y(\pi)} y^{e_y(\pi)}.$$

Furthermore, we write $\vartheta(p) = \vartheta(\{p\})$ where p is any path.

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Furthermore, we write $\vartheta(p) = \vartheta(\{p\})$ where p is any path.

Here is a basic lemma on the above.

Lemma

For any two paths π_1 and π_2, we have the following:

$$\vartheta(\pi_1 \pi_2) = \vartheta(\pi_1)\vartheta(\pi_2) \tag{3}$$

Terminology

Definition (sign classes)

The sign classes consists of the following three classes of paths, all starting at the origin.

- A **minus-path** is either the empty path or a path such that its endpoint has a negative height less than that of any other head on the path.
- A **zero-path** is a path such that its endpoint has height zero and all other heads on the path have nonnegative height.
- A **plus-path** is a path such that its endpoint has a positive height greater than that of any other point on the path.

Unique Decomposition of Paths

Theorem (path decomposition)

Let π be a lattice path over $\mathbb{Z}^2$ starting at the origin. Then π has a unique decomposition $\pi = \pi_- \pi_0 \pi_+$, where π_- is a minus-path, π_0 is a zero-path, and π_+ is a plus-path.

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Proof.

- Take π_- to be a head of π with minimal height and smallest y-coordinate.
- Let $\pi = \pi_- \sigma$ for some path σ. Then take π_+ as the head of σ with largest y-coordinate and a height of 0.
- Now clearly there exists a unique selection of π_+, which we use. This gives us a construction.
- Take ϑ of both sides of the equivalence $\pi = \pi_- \pi_0 \pi_+$ and apply our initial lemma to conclude that our construction is indeed unique.

□

Decomposition of Enumeration Function

Theorem

Let S be a step set, and let S_-, S_0, and S_+ be sets of minus-, zero-, and plus-paths with their step set being a subset of S. Then $\vartheta(S_-) = [\vartheta(S')]_-$, $\vartheta(S_0) = [\vartheta(S')]_0$, and $\vartheta(S_+) = [\vartheta(S')]_+$.

Proof.

From the initial lemma and the multiplicative nature of ϑ, we have $\vartheta(S') = \vartheta(S_-)\vartheta(S_0)\vartheta(S_+)$. The desired statement follows from the fact that the decomposition a function in $\mathbb{C}[[y, y/t]]$ is unique. □

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A Decomposition Lemma

A useful corollary of our initial lemma is the following.

Lemma (explicit decomposition)

Let $f = f(t,y) = (1 - t - t^{-1}y - z)^{-1}$, *where* $z \in y\mathbb{C}[[y]]$. *Then,*

$$f_- = [1 - t^{-1} \cdot \frac{1 - z - [(1-z)^2 - 4y]^{1/2}}{2}]^{-1} = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} t^{-k} y^{n+k} z^j \frac{k}{2n+j+k}\binom{2n+j+k}{n+k,\, n,\, j}; \quad (4)$$

$$f_0 = \frac{1 - z - [(1-z)^2 - 4y]^{1/2}}{2y} = 1 + \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} y^n z^j \frac{1}{2n+j+k}\binom{2n+j+1}{n+1,\, n,\, j}; \quad (5)$$

$$f_+ = [1 - t \cdot \frac{1 - z - [(1-z)^2 - 4y]^{1/2}}{2y}]^{-1} = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} t^k y^n z^j \frac{k}{2n+j+k}\binom{2n+j+k}{n+k,\, n,\, j}. \quad (6)$$

Examples

Example (Catalan and Ballot numbers)

Consider the step set $S = \{(0,1),(1,0)\}$. Now, $\vartheta(S') = [1 - t - t^{-1}y]^{-1}$. By the prior lemma, we evaluate $\vartheta(S_0)$ as

$$\vartheta(S_0) = \frac{1 - [1 - 4y]^{1/2}}{2y} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} y^n = \sum_{n=0}^{\infty} C_n y^n,$$

which returns the Catalan numbers C_n, and we evaluate $\vartheta(S_+)$ as

$$\vartheta(S_+) = \left(1 - t \cdot \frac{1 - z - [1-4y]^{1/2}}{2y}\right)^{-1} = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} t^k y^n \frac{k}{2n+k}\binom{2n+k}{n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_{k,n} t^k y^n,$$

returning the ballot numbers $B_{k,n}$.

Potential Extensions of the Factorization Method

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- Considering a height of 1 instead of a height of 0. This scenario was worked on by Gessel in a paper about the q-analogue of Lagrange inversion.
- Changing the definition of height to $(a, b) \to a - kb$.

Walks under Lines of Rational Slope

Introduction

A well-known problem in lattice path enumeration is the following.

Problem

Let $k = \frac{a}{b}$ *be a rational number so that* $\gcd(a, b) = 1$. *How many up-right lattice paths are there from* $(0, 0)$ *to* (bn, an) *below the line* $y = kx$ *for integers* $n > 0$*?*

Introduction

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Problem

Let $k = \frac{a}{b}$ be a rational number so that $\gcd(a, b) = 1$. How many up-right lattice paths are there from $(0, 0)$ to (bn, an) below the line $y = kx$ for integers $n > 0$?

- In the 1950s, Bizley and Grossman independently computed two formulae that give the number of paths **above** the line.
- We start with Bizley's formulation.

The Formulae of Bizley and Grossman

Theorem (Bizley, 1954)

Denote by $f(bn, an)$ the number of up-right paths from $(0,0)$ to (bn, an) weakly above $y = \frac{a}{b}x$. Then

$$f(bn, an) = \langle t^n \rangle exp \sum_{j=0}^{n} \frac{1}{(a+b)} \binom{(a+b)j}{a} t^j$$

- Bizley's proof involved using a generating function in conjunction with the cycle lemma on permutations.
- Grossman's formulation was done by casework on appearances of up-steps and right-steps, resulting in a summation over integer partitions of n.

Integer Slope Case

Theorem

Let $a_{n,k}$ be the number of walks from $(0,0)$ to (n, kn) under the line $y = kx$, where k is a fixed positive integer. Let

$$f_k(t) := \sum_{n=0}^{\infty} a_{n,k} t^n.$$

Then $f_k(t)$ satisfies the functional equation

$$t \cdot f_k(t)^k - f_k(t) + 1 = 0 \tag{7}$$

for all $t \in [0, \frac{k^k}{(k+1)^{k+1}}]$.

Integer Slope Case (proof)

We outline a probabilistic proof.

- Start at $(1, 0)$, and every second flip a coin. Suppose the coin flips heads with probability p and tails with probability $1 - p$.
- For each 'head' occurrence, move 1 unit up, and for every 'tail' occurrence, move 1 unit to the right.
- Terminate the process once the endpoint of the path is on the line $y = kx$.
- Using Chebyshev's inequality, it can be shown that the probability of the process terminating at some point is 1 when $p \in [\frac{k}{k+1}, 1]$.
- This probability is

$$\sum_{n=0}^{\infty} a_{n,k} p^{nk+1}(1-p)^n = 1.$$

Integer Slope Case (proof, cont.)

- This probability is

$$\sum_{n=0}^{\infty} a_{n,k} p^{nk+1}(1-p)^n = 1.$$

- This rewrites as

$$f(p^k(1-p)) = \frac{1}{p}$$

for all $p \in [\frac{k}{k+1}, 1]$.

- Note that $g : [\frac{k}{k+1}, 1] \mapsto [0, \frac{k^k}{(k+1)^{k+1}}]$ defined by $g(x) = x^k(1-x)$ is a bijective map.
- Thus, for all $t \in [0, \frac{k^k}{(k+1)^{k+1}}]$, $t \cdot f_k(t)^k - f_k(t) + 1 = 0$.

$\square$

Integer Slope Case

Corollary

We have the following explicit representation:

$$a_{n,k} = \frac{1}{1+nk}\binom{nk}{n}.$$

The proof is just Lagrange inversion applied to the prior theorem.

A Useful Bijection

Theorem

Let $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{c}$ for positive integers a, b, and c with $\gcd(a, b, c) = 1$. Then there is a bijection that maps the up-right walks that begin at $(0, 0)$ and stay weakly below $y = \alpha x + \beta$ to the walks that begin at $(0, b)$ that have step set $\{(1, a), (1, -c)\}$ and are above the x-axis.

Lattice Paths under Lines of Slope 2/5

Theorem

Consider walks in $\mathbb{N}^2$ with step set $\{-2, +5\}$. The number of such walks starting at altitude 3 and ending at altitude 0 is given by the generating function $F(x)$, and altitude 4 and ending at altitude 1 is given by the generating function $G(x)$. Let $P(z) = z^{-2} + z^5$ and the kernel $1 - xP(z) = 0$ have small roots (for z) of $r_1(x)$ and $r_2(x)$. We have

$$F(x) = -\frac{r_1 r_2 (r_1^4 - r_2^4)}{x(r_1 - r_2)}$$

and

$$G(x) = \frac{r_1^6 - r_2^6}{x(r_1 - r_2)}.$$

Lattice Paths under Lines of Slope 2/5

Theorem

Denote A_n and B_n by the number of paths beginning at the origin and ending at $(5n-1, 2n-1)$ that respectively stay weakly below $y = (2/5)x + 1/5$ and stay weakly below $y = (2/5)x$; then we have

$$A_n + B_n = \frac{2}{7n-1}\binom{7n-1}{2n}.$$

Lattice Paths under Lines of Slope $2/5$

Theorem

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Remark. A_n and B_n have rather ugly forms, and it is fairly surprising as to why their sum is so nice! There are a number of factors that contribute to this result.

Conclusion

In conclusion, we surveyed a number of topics in the modern combinatorial field of lattice path enumeration. In particular, we looked upon:

- kernel methods for various models;
- finite and infinite group models for the algebraic kernel method;
- Laurent series factorization and a combinatorial counterpart;
- counting walks under lines of rational slope, in particular the Grossman-Bizley formulae, integer slopes, and the case when the slope is $2/5$.

Thanks for attending my talk!