The Hahn-Banach Theorems

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Infinite Dimensional Vector Spaces and Norms

When talking about vector spaces, there are cases where finite dimensions are not enough to describe what we want.
Defining Norms: A norm is a way to define the size of a vector in a vector space.

Definition 1: Norm

A norm $\|\cdot\|$ is a map from a vector space V to a field of scalars $\mathbb{K}$ such that

- $\|\lambda v\| = \lambda\|v\|$ for all $v \in V, \lambda \in \mathbb{K}$
- $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$
- $\|x\| = 0$ if and only if $x = 0$

Dual Spaces and Functionals

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Definition 3

The dual space of a vector space V is the vector space of all functionals. It Is denoted with a V^*

Hahn Banach Extensions

Extending Fuctionals

Hahn-Banach

Let V be a normed vector space with V_0 a subspace. Let f be a continuous linear functional on V_0. Then there exists a continuous linear functional F from V to $\mathbb{R}$ such that F extends f and

$$\|f\| = \|F\|$$

Overview of Proof

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To do this, we extend f from V_0 to $V_0 + \mathbb{K}x$
Then, using Zorn's lemma we show that there exists a maximal extension.

Overview of proof cont'd ...

Proof of Maximal element
We define chains in V for the purpose of this proof by pairings (V_n, f_n) where $V_n \supseteq V_0$ and f_n extends f.

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We then define a partial ordering on these pairings by $(V_n, f_n) \leq (V_m, f_m)$ is $V_n \subseteq V_m$ and f_m extends f_n.

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We then define an upper bound for these chains by (V_{max}, g) by

$$V_{max} = \bigcup_n (V_n) \text{ and } g \text{ extends all } f_n\text{s}$$

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Now, by Zorns lemma, we have a maximal element (V, F), which by its maximality extends f to all of V

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However, there are theorems between Axiom of choice and Hahn-Banach. For example, Banach-Alaoglu,Ultra filter lemma, etc ...

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Banack Tarski Theorem

A three-dimensional Euclidean ball can be finitely cut and rearranged into two copies of itself.

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However, there are theorems between Axiom of choice and Hahn-Banach. For example, Banach-Alaoglu,Ultra filter lemma, etc ...
Hahn-Banch is one of the weakest theorems that can be used to prove Banach-Tarski.

Well defined functionals in Dual spaces

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- This leads to some interesting results. Namely, that the map from V to V^{**} is isometric.

Relation to Measure

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Definition 4

μ is a measure on the $\sigma - algebra$ of X if

- for all A in the sigma algebra, $\mu(A) \geq 0$, with $\mu(\emptyset) = 0$
- for all countable collections of disjoint $\{E_n\}_{n=1}^{\infty}$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Using Hahn-Banach one can construct a non-measurable set. Which is how Hahn-Banach implies Banach-Tarski

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Geometric Hahn-Banach Theorem

Let X be a topological vector space over $\mathbb{R}$, N a linear subspace of X, and O a non-empty open convex subset of X such that N $\cap$ O $= \emptyset$, then there exists a closed hyperplane H of X such that

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This gives us a way to extend a separation between O and N to separation between O and a hyperplane.

Hahn-Banach Separation Theorems cont'd...

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Separation of two convex sets

If A is an open subset of X and B is a subset of X, then there exists and separating function $p : X \to \mathbb{R}$ such that there exists an k such that $p(a) < k \leq p(b)$ for all $a \in A, b \in B$

This also further implies that any two points, if one can find an open convex set around them, can be separated.
Furthermore, if B is a cone we can set $k = 0$.

The Weak Topology

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The weak topology gives us a nice property for a vector space.
Namely, that the space is locally convex.
With this, we get the following property about vector spaces.

Proposition 1

A vector space under the weak topology is a Hausdorff space.

Hausdorff Topology

A Hausdorff topology is a topology where any two points $x, y \in X$ have neighborhoods U, V such that $U \cap V = \emptyset$

The moment problem

Definition 5

The nth moment m_n of a measure μ be a non-negative Borel measure defined on $\mathbb{R}$ is defined by

$$m_n^\mu = \int_{\mathbb{R}} x^n \mu(dx)$$

This gives us a way to define moments in terms of a measure. The moment problem asks the same question in reverse.

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This gives us a way to define moments in terms of a measure. The moment problem asks the same question in reverse. Let C be a closed subset of $\mathbb{R}$ with $m := (m_n^\mu)_{n=0}^\infty$. Does there exist a measure μ such that the moment of the measure is m where a m_n is defined as

$$m_n = \int_C x^n \mu(dx) \text{ for all } n \in \mathbb{N}$$

Moment problem cont'd...

We can rephrase the moment problem in terms of linear functionals by rephrasing the moments as linear functionals.

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Moment Problem With Linear functionals

Let C be a closed set of $\mathbb{R}$ an $L : \mathbb{R}[x] \to \mathbb{R}$. Does there exist a measure μ such that

$$L(p) = \int_{\mathbb{R}} p(x)\mu(dx) \text{ for all } p \in \mathbb{R}[\mathrm{x}]$$

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Now that we have the problem in terms of linear functionals. This allows us to use Hahn-Banach and change the problem to ask something different.

New moment Problem

New version of the moment Problem

Let L : $\mathbb{R}[x_D] \to \mathbb{R}$ linear.If M_S is archemidean then, there is a K_S-representing measure μ for L if and only if $L(M_S) \geq 0$.

For this version of the problem
K_S is $\{x_D \in \mathbb{R}[x_D] : g_i(x_D) \geq$
0 for all g_i in a finite set of polynomials $S\}$
and M_S is $\{\sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[x_D]^2, i = 1, 2, 3 \ldots s\}$

Questions

Any Questions?

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