Proving the Bonnet Myers Theorem

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Introduction

Notations of Multi-Linear Algebra

Definition

Tensor products of a vector space V are multi-linear maps, meaning that if we multiply any of the elements $V_1 \otimes V_2 \otimes \ldots \otimes V_r \to \mathbb{R}$ by a scalar, the result $V_1^* \otimes V_2^* \otimes \ldots \otimes V_r^*$ (i.e the tensor product) will also be multiplied by the same scalar.

Definition

A (r,s) tensor field in V is given by the set of elements of the tensor products where V_s^r r times of $V \otimes \ldots \otimes V$ multiplied by s times of $V^* \otimes \ldots \otimes V^*$.

Riemannian Manifolds

A Riemannian manifold (M,g) is a form of smooth manifold which admits a Riemannian metric.

Definition

A Riemannian metric is a (0,2) tensor field g$\in$ T^2(M) such that for all p$\in$ M, g_p is a bilinear, symmetric and positive definite (g_p(v, v) > 0 for every v $\neq 0$, v $\in$ T_pM) inner product on T_pM.

Every smooth manifold admits at least one Riemannian metric.

Levi Civita Connection

Definition

A connection is a map on the vector bundle $\pi\colon E \to M$ where

$$\nabla\colon \mathfrak{X} \times \Gamma(E) \to \Gamma(E)$$

. $\nabla_v T$ is defined as the connection on M for all v $\in \mathfrak{X}$(M) and S,T $\in \Gamma(E)$. An affine connection is essentially a map between two neighbouring points in a tangent space.

Definition

The Levi-Civita Connection is a unique, symmetric and torsion free connection that can be induced on any manifold (M,g) which satisfies the following conditions:

1. dg(X,Y) = g(DX, Y) + g(X, DY)
2. $\nabla_X Y - \nabla_Y X =$ [X,Y]

Completeness

1. If and only if its metric space is complete (Hopf-Rinow)
2. Its exponential map $\exp_p$ can be defined on the entire TM for all p $\in$ M and therefore γ can be extended between $(-\infty, \infty)$
3. Any two points, say p, q, can be connected by a length minimising geodesic whose distance d(p,q) is the minimum length of all possible curves in the direction p to q.
4. All bounded closed subset in M are compact.

Riemannian Distance

The Riemannian distance is the shortest length (also called a geodesic) of a curve between two points on a Riemannian manifold.

Definition

We can define the Riemannian distance between two points p,q by setting a piecewise smooth curve $\gamma\colon [a,b] \to M$ with the tangent vector $T_{\gamma t}M$ given by $\dot{\gamma}(t) = d\gamma\frac{d}{dt}$ for all t$\in [a,b]$. The length of the curve, L(γ), can be written as

$$L(\gamma) = \int_a^b \sqrt{< \dot{\gamma}(t), \dot{\gamma}(t) >_{\gamma(t)}}\, dt$$

. If L(γ) = d(p,q), we call the curve length minimising.

Sectional Curvature

Definition

For a point p$\in$ M and a 2-dimensional subspace or plane $\Pi \subset T_pM$, we define the sectional curvature of Π at p as

$$K(\Pi) = K(x, y) \quad (1)$$

$$:= \frac{R_m(x, y, y, x)}{g(x, x)g(y, y) - (g(x, y))^2} \quad (2)$$

where x,y are vectors $\in \Pi$ which form the basis and Rm denotes the Riemann curvature tensor.

Properties of the sectional curvature:

1. K(x,y) is independent of the choice of x, y $\in \Pi$
2. If the vector basis {x,y} is orthonormal, then $K(\Pi) = R_m(x, y, y, x)$

Ricci Curvature

Definition

The Ricci curvature tensor is defined by

$$Ric(X, Y) := tr(R(\cdot, X)Y)$$

. tr denotes a trace, which is simply a contraction map.

Properties of the Ricci curvature:

1. In local coordinates, the Ricci curvature tensor can be written as
$$Ric = Ric_{ij} dx^i \otimes dx^j$$
2. Since R is symmetrical (i.e Ric(X, Y) = Ric(Y, X) for all X,Y $\in \mathfrak{X}$, tracing through any of the two arguments returns a result of either Ric or 0.

Bonnet Myers Theorem

Theorem

The Bonnet Myers theorem states that for any complete Riemannian manifold (M^n,g) whose sectional curvature, sec(M) $> \delta$, where δ is a positive constant, its Ricci curvature, R, satisfies:

$$Ric(M) \geq \delta(n-1)$$

. We can then estimate its diameter, diam(M), since it is always bounded by

$$sup_{p,q \in M}\, dist(p,q) \leq \frac{\sqrt{\pi}}{\delta}$$

. This is sufficient to show that M is compact.

-Cheng later proved in his rigidity theorem that all manifolds which satisfy the Bonnet Myers theorem have a constant sectional curvature k.

Outline of the Proof

-Assume that there exists a R which satisfies dist(p,q) $\leq \frac{\sqrt{\pi}}{\delta}$
-We also assume that there exists a geodesic γ between points p,q that is defined on the interval [0,L]such that $|\gamma^{'}| = 1$ after re-parametrisation.
-set a parallel unit vector field to γ (i.e $< W, \gamma^{'} >= 0$) and V(t)= $sin(\frac{\pi t}{L})$
-$\mathrm{L}_0^{'} = 0$ since γ is a geodesic
-Using the second variation of arc length equation

$$-\int_0^L < V^{''} - R(V, \gamma^{'})\gamma^{'}, V > dt$$

We can prove that $L(x(s, \cdot)) = L_x(s) \leq L_x(0) = L(\gamma)$ if s is infinitesimally small. Therefore, we can show that length of the curve x(s,·) is actually shorter than γ between p,q $\Rightarrow$ contradicting!