Proving the Bonnet-Myers Theorem

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Abstract

The aim of this paper is to provide an introduction to the Bonnet-Myers theorem, which relates the Ricci curvature to the topology of a manifold. We begin by introducing the basic concepts of manifolds and tangent spaces, and proceeds to define Riemannian manifolds and their properties such as the Levi-Civita connection. Geodesics and the curvature of a Riemannian manifold is also explored, focusing specifically on the sectional and Ricci curvatures. Finally, we present a proof of the Bonnet-Myers theorem.

1 Introduction

One of the fundamental aspects of Riemannian geometry is the use of comparison theorems, which establish relationships between the geometry of a given manifold and simpler reference spaces, such as spaces of constant curvature. These theorems allow for comparisons of quantities such as lengths of curves, volumes of regions, and sectional curvatures. By comparing the geometry of a manifold to a reference space, comparison theorems provide valuable insights into the intrinsic curvature and global properties of Riemannian manifolds.

2 Preliminaries

2.1 Smooth Manifolds

A topological space M is called an n-dimensional manifold if M is

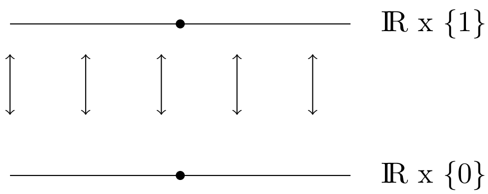
1. locally Euclidean at any point ((i.e every point in the topological space has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$)
2. Haussdorf

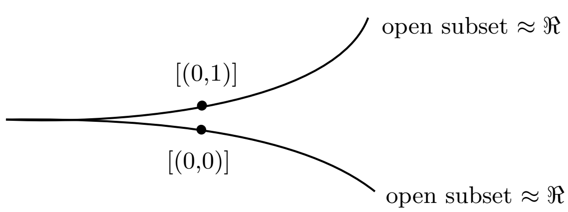
3. second countable

We now explain the significance of each of the above criteria by providing non-examples.

The Haussdorf property states that for any two points p,q $\in$ M, there exists a neighbourhood of p and q such that $U_p \cap U_q \neq \emptyset$. The Haussdorf condition ensures that the curve does not branch and that two distinct points do not have a Euclidean distance of 0.

Case 1: a manifold is non-Haussdorf
Example: We first consider two real lines in standard topology X= ($\mathbb{R}$ x {0,1})/$\sim$ where (x,0) $\sim$ (x,1) if x< 0.





By projecting the quotient map π: $\mathbb{R}$ x {0,1} $\rightarrow$ X such that it assigns every point in the space the corresponding equivalence class and assuming that A$\subset$ X is open if and only if π^{-1}(A) is open, we show that the two origins are the first points which are not identified together. Hence, each branch of the projection map denotes a open subset, making it impossible to find separating neighbourhoods around these two origins.

Theorem 2.1. *A non-Hausdorff manifold admits no partition of unity.*

Proof. For a partition of unity to exist, we require a function to be 0 outside of every open set of the manifold. From the above example, we have shown that the two adjacent points [(0,0)] and [(0,1)] cannot belong to the same coordinate chart. Therefore, there exists open sets containing one but not the other. □

Definition 2.1. B ⊂ P(X) is called a basis for the topology on X if for every A ⊂ X, A is open :

- ⇔ A is a union of elements in β
- ⇔ ∀ p ∈ A ∃ B ∈ β such that p ∈ B ⊂ A

We can now define the topological basis as B={B(x,r) ⊂ ℝ | x∈ Q^n, r∈ Q, r>0}. If a countable basis exists for a manifold, we refer to the manifold as second-countable.

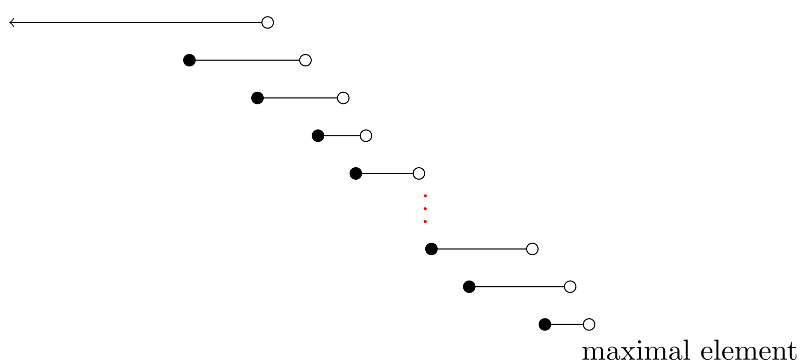
Remark 2.2. second-countable ⇒ first-countable

Lemma 2.3. *There exists an uncountable, well-ordered set S such that S has a maximal element $\Omega \in S$ and that the set $\{x \in S \mid x$ ¡ $\alpha\}$ is countable for all $\alpha \neq \Omega$*

The second countable property ensures that a manifold can be embedded within some finite dimensional Euclidean space, which we will later show through Whitney's Embedding Theorem.

Case 2: a manifold is non second-countable

We provide a non-example of the second-countable property using the set X:= (-∞, 0) S x [0, 1). We assume that a manifold M is non second-countable, i.e S is an uncountable subset with an infinite number of open set (X:= S x ℝ)



For a randomly chosen point in the set, there is only finitely many number of intervals before it. This shows that while each individual subset is homeomorphic, the whole set is not homeomorphic to ℝ. Therefore, the resulting space is not second countable as we can easily find an uncountable disjoint family of open sets in A.

Theorem 2.4. *Let (x, τ) be a second-countable space where $A \subseteq X$. Then, there exists at least one accumulation point $a \in A$.*

Proof. (by contradiction) We initially assume that an uncountable set A has no accumulation points and therefore every point is isolated.
If this holds true, then there exists an open neighbourhood U_x of x for each $x \in A$ such that no points in A and x are different.

$$U_x \cap A = \{x\} \tag{1}$$

□

Definition 2.2. A topological space X is defined as locally Euclidean at a point p$\in$ X if there is an open neighbourhood p$\in$ U $\subset$ X that is homeomorphic to both $\mathbb{R}$ some open subset of $\mathbb{R}$.

Proposition 2.5. The locally Euclidean property holds iff a topological space satisfies either of the following criteria :

- every point on the space has a neighbourhood homeomorphic to $\mathbb{R}^n$
- the neighbourhood should also be homemorphic to an open ball of $\mathbb{R}^n$

2.2 Charts and Atlases

Definition 2.3. A coordinate chart on M is a pair of (U, ϕ) where U $\subset$ M is an open subset and ϕ: U $\rightarrow$ = ϕ(U) $\subset$ $\mathbb{R}$ is homeomorphic to an open subset.

$$\phi(p) = (\phi^1(p), ...\phi^n(p)) = (x^1(p), ...x^n(p))$$

where x denotes the component functions of ϕ

Definition 2.4. Two manifolds M and N are called diffeomorphic if there exists a smooth bijective map M $\rightarrow$ N which has a smooth inverse $\psi \circ$ f $\circ$ ϕ^{-1}.

Definition 2.5. As all smooth manifolds admits a C^∞ differential structure, we can define d = dim M is the dimension of the manifold M.

Definition 2.6. A C^∞ differentiable structure on M is a collection of coordinate charts ϕ_α : $U_\alpha \rightarrow V_\alpha \subseteq \mathbb{R}^d$ such that

(i) M = $\bigcup_{\alpha \in A} U_\alpha$

(ii) for every α, β, the change of local coordinates is a smooth C^∞ map on its domain $\phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^d$. Any changes in local coordinates therefore marks a diffeomorphism between the two open subsets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$.

(iii) the collection of charts ϕ_α is maximal, meaning that ϕ is included in the collection if a chart ϕ of M is compatible with all ϕ_α. This follows from condition (ii).

Definition 2.7. An atlas is defined as a collection of charts in a manifold M. A= $(\{U_a, \phi_a\} : a \in I)$ such that the union of these charts cover M.

Definition 2.8. A transition map compares two charts in an atlas, (U_α, ψ_α) and (U_β, ψ_β) where $U_\alpha \cap U_\beta$ is non-empty set by comparing the composite of one chart with the inverse of the other. The transition map from ψ_α) to ψ_β) is given by $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha (U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$.

Definition 2.9. For two connected smooth manifolds, M and $\tilde{M}$, a map $\pi \colon \tilde{M} \to M$ is considered a smooth covering map if all the conditions below hold true:

(i) the covering space, $\tilde{M}$, is globally and locally path connected

(ii) π is continuous, surjective and every element of $\pi^{-1}(U)$ can be diffeomorphically mapped onto U by π where U is a neighbourhood around a point p$\in$ M.

Definition 2.10. A universal covering space is defined as a map between a path-connected and simply connected covering space.

3 Tangent Spaces

3.1 Tangent Spaces and the Differential of a Smooth Map

Let $M \subset \mathbb{R}^n$ be an embedded manifold and let there be a tangent space to M at the point p$\in$ M. We can consider a tangent space as a set of directional derivatives. Given a point p$\in$ M, the space C^∞(p) is the set of functions f defined on an open neighbourhood U of p such that f : $U \to \mathbb{R}$ is smooth at P.

Definition 3.1. A tangent vector X at any point p$\in$ M is a linear functional i.e a smooth map

$$X : C^\infty(M) \to \mathbb{R}$$

which satisfies Leibniz' product rule

$$X(fg) = Xf \cdot g(p) + f(p) \cdot X_g$$

A tangent vector must satisfy this rule since it forces the linear operator δ:f $\mapsto \delta$(f) $\in \mathbb{R}$ such that f only depends on the first order of approximation of p. The tangent space to M at p, TpM is the collection of all tangent vectors.

Proposition 3.1. T$_p$M is a vector space with the same dimensionality n as the manifold M.

Proof. From definition 3.1, we already know that the tangent space T$_p$M is the set of equivalence classes of smooth curves, $[\gamma]$, where γ is defined on some open interval containing 0 such that $\gamma(0)$=p.

Let (U,ϕ) be a chart at p $\in$ M. We now consider a map that sends a curve γ : (-ϵ, ϵ) $\to$ M through the point p where $\gamma(0)$=p to its corresponding tangent vector $(\phi \circ \gamma)'(0) \in \mathbb{R}$. This induces a new map $\bar{\phi}$: T$_p$M $\to \mathbb{R}$.

Since $\bar{\phi}$: T$_p$M $\to \mathbb{R}^n$ is clearly a bijection following the definition of an equivalence class, we are now left to show that the tangent space is a vector space by applying the addition and scalar multiplication operations, which makes F$_{\phi,p}$ linearly isomorphic to the two vector spaces.

1. addition: $[\gamma_1 + \gamma_2]$= F$^{-1}_{\phi,p}$ (F$_{\phi,p}[\gamma_1] + F_{\phi,p}[\gamma_2]$)
2. scalar multiplication: c$[\gamma]$=F$^{-1}_{\phi,p}$(c $\cdot$ F$_{\phi,p}[\gamma]$) where c is a constant

This completes the proof. □

Definition 3.2. Since a chart (U,ϕ) defines a local coordinate system (x$^1 \ldots x^n$) of M, a set of coordinate vectors for a chart (U,ϕ, x^i) at any point p$\in$ M is given by

$$\frac{\delta}{\delta x^i}\Big|_p g = \frac{\phi^{-1} \circ \delta g}{\delta x^i}\Big|_{\phi(p)}$$

where g$\in C^\infty(p)$ and $\frac{\partial}{\partial x^1}\Big|_p \ldots \frac{\partial}{\partial x^n}\Big|_p$ forms the basis of T$_p$M.

Definition 3.3. The tangent bundle of a manifold M, represented by TM, is the set of disjoint union of all tangent spaces at each point in M.

$$TM := \bigcup_{p \in M} T_p M$$

. A projection map,

$$\pi : TM \to M, T_pM \ \exists X \to p$$

is surjective and is the inverse of the tangent space at point p (T_pM x {p} $= \pi^{-1}(p)$). Since the tangent bundle is smooth, π is also smooth.

Proposition 3.2. If dim M= n, then dim TM = 2n.

Proof. Let $\{(U_\alpha, \phi_i)\}_{i \in I}$ be an atlas of M consisting of the compatible charts (U_i, ψ_i) and (U_j, ψ_j) and let π denote the natural projection. Let $V_i = \pi^{-1}$ be an open set and $\psi_i : V_i \to \mathbb{R}^n$ x $\mathbb{R}^n = \mathbb{R}^{2n}$. We can re-write ψ_i (p,X) as

$$\phi_i(p, X) = \phi_i(p), d(\psi_i)_p(X))$$

, which is a continuous bijection onto the open set with a continuous inverse, a property which also holds for ψ_i .

We then compare the two charts by the transition map $\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j)$ x $\mathbb{R}^n \to \phi_j(U_i \cap U_j)$ $\mathbb{R}^n$. This allows us to obtain

$$\begin{aligned} \psi \circ \psi^{-1}(q, u) &= (\phi_j \circ \phi^{-1}(q), d(\phi_j)_{\phi^{-1}(q)} \circ d(\phi_i)_q^{-1}(u)) \\ &= (\phi_j \circ \phi^{-1}(q), d(\phi_j \circ \phi_i^{-1})_q(u)) \end{aligned}$$

, which is clearly a diffeomorphism since it is a smooth bijection with a smooth inverse. We have now shown that the tangent bundle satisfies all the conditions of a 2n manifold. □

Definition 3.4. A cotangent space at the point p $\in$ M, T_p^*M, is isomorphic to the dual of the tangent space T_pM. We refer to the elements contained in the cotangent space as the cotangent vectors.

Definition 3.5. The cotangent bundle, T∗M, is the disjoint union of all cotangent spaces at all points of M.

$$T * M = \{(p, \omega) | p \in M, \omega \in T_p^*M\}$$

The natural projection is a map π : T∗M $\to$ such that π(p, ω). Since the cotangent bundle is a smooth function, π is also smooth.

Lemma 3.3. *A cotangent bundle is a vector bundle.*

3.2 Vector Fields

Definition 3.6. The identity map on M is a bijective map which satisfies $(\mathrm{Id}_M)_{*p} = \mathrm{Id}_{T_pM} :$ $\mathrm{T}_p\mathrm{M} \to \mathrm{T}_p\mathrm{M}$ for all p $\in$ M. This means that each point is returned to the same point under parallel transport id : (M, g) $\to$(M, g).

Definition 3.7. A vector field on M is a smooth, linear map over C^∞ X : M $\to$ TM such that the composite function πX is the identity map. The set of all vector fields on M is given by $\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is a commutative vector space. We can define the gradient of the vector field by (gradf(p),v) = df_p(v) for all p $\in$ M and v $\in$ T_pM.

Definition 3.8. A vector bundle with a rank l where l $\geq$ 0 is a triple (E, M, π). A vector bundle is a total space E where E has dimensions n+l with a base space of M and a smooth surjective map π : E $\to$ M . Any fibre π^{-1}(p) is isomorphic to V for all p $\in$ M.

Remark 3.4. Properties of a Vector Bundle:

(i) A vector bundle has a locally trivial fibration

(ii) For any point p$\in$ M and $\alpha \in$ A,

$$\psi_\alpha(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^l$$

Subsequently,

$$\psi_\alpha\Big|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \mathbb{R}^l$$

shows a l-linear transformation and is therefore isomorophic on the vector space.

3.3 Notations of Multi-Linear Algebra

Definition 3.9. Tensor products of a vector space V are multi-linear maps, meaning that if we multiply any of the elements $\mathrm{V}_1 \otimes \mathrm{V}_2 \otimes \ldots \otimes \mathrm{V}_r \to \mathbb{R}$ by a scalar, the result $\mathrm{V}_1^* \otimes \mathrm{V}_2^* \otimes \ldots \otimes \mathrm{V}_r^*$ (i.e the tensor product) will also be multiplied by the same scalar.

Remark 3.5. The tensor product operation is linear, non-commutative and associative.

Definition 3.10. A (r,s) tensor field in V is given by the set of elements of the tensor products where $\mathrm{V}_s^r :=$ indicates r times of V $\otimes \ldots \otimes$ V multiplied by s times of $\mathrm{V}^* \otimes \ldots \otimes \mathrm{V}^*$.

Definition 3.11. For some v_1, $v_2 \ldots v_k$ in V and W^1, $W^2 \ldots W^s \in V^*$, a reducible tensor $T \in V^r_s$ is one that can be written in the form $T = v_1 \otimes v_2 \otimes \ldots \otimes v_r \otimes W^1 \otimes W^2 \otimes \ldots \otimes W^s$.

4 Riemannian manifolds

We begin by defining an inner product on a vector space.

Definition 4.1. An inner product on a vector space V is a function $<\cdot, \cdot>$: VxV $\to$ F which associates a pair of smooth contravariant vector fields to a scalar vector field $<X, Y>$. An inner product satisfies the following properties of symmetry, bilinearity and non-degeneracy:

Definition 4.2. A smooth manifold M with a smoothly varying inner-product is called a Riemannian manifold. All Riemann manifolds are path-connected, second countable and Haussdorf by definition.

Definition 4.3. A Riemannian metric is a (0,2) tensor field $g \in T^2(M)$ such that for all $p \in M$, g_p is a bilinear, symmetric and positive definite ($g_p(v, v) > 0$ for every $v \neq 0$, $v \in T_pM$) inner product on T_pM.

We can define the Riemannian metric using local-coordinates. We first consider the functions

$$g_{ij} : U \to \mathbb{R}^n$$

and

$$g_{ij}(p) := g_p(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p)$$

given a Riemannian manifold (M,g) and a chart (U, ϕ, x^i). The functions g_{ij} are the components of g with respect to $\{x^i\}$ since for each $p \in U$, $(g_{ij}(p))^n_{i,j=1}$. We can then apply a local-coordinate frame $\{dx^i\}$ on the cotangent bundle T*M, which gives us

$$g = g_{ij}dx^i \otimes dx^j$$

. However, as g and hence g_{ij} are symmetric, we can re-write the above formula as

$$\frac{1}{2}(g_{ij}dx^i \otimes dx^j + g_{ji}dx^i \otimes dx^j)$$

.

Definition 4.4. (Parition of Unity) An open cover $\{V_\alpha\}_{\alpha \in A}$ is called locally finite if, for every p$\in$ M, there exists a neighburhood U of p such that the set

$$\{\alpha \in A | U \cap V_\alpha \neq 0\}$$

is finite.
The support of a function u: M $\rightarrow$ $\mathbb{R}$ is defined to be

$$supp(u) = \overline{\{p \in M | u(p) \neq 0\}}$$

Given a locally finite open cover of M, $\{V_\alpha\}_{\alpha \in A}$, a partition of unity subordiante to $\{V_\alpha\}_{\alpha \in A}$ is a collection $\{u_\alpha\}_{\alpha \in A}$ of smooth functions u_α: $V_\alpha \rightarrow$ [0,1] such that

$$\sum_{\alpha \in A} u_\alpha(p) = 1$$

.

Proposition 4.1. Every smooth manifold (M,g) admits at least one Riemannian metric.

Proof. Let M be a differentiable manifold and let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a locally finite atlas such that $U_\alpha \subseteq$ M is an open subset and $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha\ (U_\alpha)$ $\subseteq \mathbb{R}$ be diffeomorphisms. We then choose a Riemannian metric g_α on each U_α.

$$g_\alpha = \sum dx_a^i \otimes dx_a^i \tag{2}$$

Let $\{T_\alpha\}$ be a partition of unity subordinate on the chosen covering $\{U_\alpha\}$. We define g as

$$g = \sum_\alpha \rho_\alpha g_\alpha \tag{3}$$

□

This is a finite sum in the neighbourhood of each point. As it must be a positive definite since for any $\rho \in$ M, there always exist some α such that ρ_α(p) $\geq$ 0. So it is a Riemannian metric on M.

Definition 4.5. Let f: M $\rightarrow$ N be a smooth map between two distinct manifolds and let g be the Riemannian metric on N. The pullback metric, $(\phi * g)$ on N, can be defined as $(\phi * g)_p(u,v) = g_{\phi(p)}(d\phi_p(u), d\phi_p(v)$ for all p $\in$ N and u,v $\in$ T_p(N). The pullback metric only exists if ϕ is a local diffeomorphism.

5 The Levi-Civita Connection

We must first understand the idea of connections and affine connections along vector bundles as well as the covariant derivative before proceeding to define the Levi-Civita Connection.

Definition 5.1. A connection is a map on the vector bundle $\pi \colon E \to M$ where

$$\nabla \colon \mathfrak{X} \times \Gamma(E) \to \Gamma(E)$$

. $\nabla_v T$ is defined as the connection on M for all v $\in \mathfrak{X}$(M) and S,T $\in \Gamma(E)$ if it satisfies the following conditions:

(i) The map v$\mapsto \nabla_v T$ must be linear over C^∞(M) for every T $\in \Gamma(E)$

$$\nabla_{f_1 v + f_2 w} T = f_1 \nabla_v X + f_2 \nabla_w T$$

(ii) The map T$\mapsto \nabla_v T$ must be linear over $\mathbb{R}$ for every v $\in \mathfrak{X}$(M)

$$\nabla_v (g_1 S + g_2 T) = g_1 \nabla_v S + g_2 \nabla_v T$$

for any g_1, $g_2 \in \mathbb{R}$

(iii) It satisfies the product rule: $\nabla_v (f_1 T) = v f_1 \cdot T + f_1 \nabla_{f_1} T$

Note that f_1, f_2 are simply scalar multiples in C^∞(M).
More specifically, an affine connection is a connection on the tangent bundle TM such that E(M) = $\mathfrak{X}(M)$.

Definition 5.2. The covariant derivative enables us to differentiate tensor fields along vector fields on curved manifolds as it is independent to the local frame $\{e_i\}$ and satisfies conditions (ii) and (iii) from definition 5.1. The covariant derivative is denoted by

$$D_t = \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

and can also be written locally as

$$D_t V\Big|_{t_0} = V^i(t_0) \nabla_{\dot{\gamma}(t_0)} e_i \Big|_{\gamma(t_0)} + (V^i)'(t_0) e_i \Big|_{\gamma(t_0)}$$

Hence, if V can be extended over M, we can write D_tV(t) = $\nabla_{\dot{\gamma}(t)} V'$ where V $\in \Gamma(\gamma)$ for the extension V'.

Definition 5.3. Let $\gamma\colon (a,b) \to M$ be a curve and $X_0 \in T_{\gamma(t_0)}M$. The map $P^{\gamma}_{t_0,t} \to T_{\gamma(t)M}$ where $X_0 = X(\gamma(0)) \mapsto X(\gamma(t))$ is defined as the parallel transport along γ from $\gamma(t_0)$ to $\gamma(t)$. The vector field V along the starting point, $X\gamma(t_0)$ is parallel to γ such that $X_0 = X\gamma(t_0)$.

Lemma 5.1. *A parallel transport is always a linear isomorphism.*

Definition 5.4. (Levi Civita Connection) The Levi-Civita Connection is a unique, symmetric and torsion free connection that can be induced on any manifold (M,g) which satisfies the following conditions:

(i) dg(X,Y) = g(DX, Y) + g(X, DY)

(ii) $\nabla_X Y - \nabla_Y X =$ [X,Y]

Proof. Here, we offer a very succint proof of the existence and uniqueness of the Levi Civita connection and affine connections in general. Let X, Y, Z be distinct fields that are defined by

(i) X$< Y, Z >= < \nabla_X Y, Z >, Z, \nabla_X Z >$

(ii) Y$< X, Z >= < \nabla_Y Z, X >, Z, \nabla_Y X >$

(iii) Z$< X, Y >= < \nabla_Z X, Y >, Z, \nabla_Z Y >$

By (i)+(ii)-(iii), we obtain the following equation:

$$X < Y, Z > +Y < X, Z > +Z < X, Y > = 2 < \nabla_Y X, Z > + < [X, Y], Z > \\ + < [X, Z], Y > + < [Y, Z], X >$$

□

Definition 5.5. (Defining the Christoffel symbols) Let ∇ be the Levi-Civita connection on M, and let $\phi\colon U \to V$ be a coordinate chart with the local frame $\{x_i\}$ coordinates. We can define the Christoffel symbols with respect to the coordinate frame $\{\frac{\delta}{\delta x^i}\}$ where its covariant derivative = D$\{\frac{\delta}{\delta x^i}\}$. Setting X, Y, Z in the directions $\frac{\delta}{\delta x^i}$, $\frac{\delta}{\delta x^j}$ and $\frac{\delta}{\delta x^k}$ respectively, we can re-write equation (7) as

$$2g_{kl}\Gamma^l_{ij} = \frac{\delta g_{jk}}{\delta x^i} + \frac{\delta g_{ik}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^k}$$
$$\Rightarrow \Gamma^k_{ij} = \frac{1}{2} g_{kl}(\frac{\delta g_{jk}}{\delta x^i} + \frac{\delta g_{ik}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^k})$$

. Γ^k_{ij} is defined as the Christoffel symbols for a connection on M.

6 Geodesics

Definition 6.1. A geodesic is a curve $\gamma\colon (a, b) \to M$ whose covariant derivative $\mathrm{D}_t\dot{\gamma}(t)$ =0 and has zero acceleration $\ddot{\gamma} \equiv 0$. A geodesic is the shortest arc between the two points a,b.

Theorem 6.1. *For any $p \in M$ and $v \in T_pM$, there exists a a unique geodesic $\gamma\colon (T_-, T_+) \to M$ where (T_-, T_+) is a maximal interval $\subseteq \mathbb{R}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.*

Remark 6.2. $\dot{\gamma}$ denotes the velocity of a curve.

Lemma 6.3. *(Rescaling Lemma) Assume a geodesic $c\colon [0, a] \to M$ where k > 0. A curve γ can be defined by $\gamma\colon [0, \frac{a}{k}]$ and therefore $\gamma(t) = c(kt)$.*

Definition 6.2. The exponential map, $\exp_p$, is a smooth map such that

(i) $\exp_p\colon \epsilon_p \to M$ where ϵ is an open subset of T_pM

(ii) For a unique geodesic with starting point p and initial velocity v, $\gamma_v\colon [0, 1] \to M$ such that $\gamma_v(0)$=p and $\gamma_p'(0)$, it can be defined as $\exp_v = \gamma_v(1)$

(iii) There exists at least one $\epsilon > 0$ for every p$\in$ M such that the exponential map is a diffeomorphism onto itself

$$exp_p\colon \{X \in T_pM||X| < \epsilon\}$$

Proof. We can prove condition (i) using the rescaling lemma. By setting t=1, we obtain

$$exp(cX) = \gamma_{cX}(1) = \gamma_X(c)$$

. The proof of (ii) follows in the same way as (i) since $\gamma_{cX}(t) = \gamma_X(ct)$.
Finally, to prove (iii), we aim to prove that the exponential map $\exp_p\colon U \to V$ is a diffeomorphism, where U and V are two open neighbourhoods such that U $\subset T_pM$ and V$\subset M$ for every p $\in M$ and p $\in V$.

We can now apply the Inverse Function Theorem and subsequently use the fact that the differential of the exponential map at the origin $(\exp_p)_{*0}$ cannot be inverted to show that this is equivalent to the identity map I_dM. □

To apply the Bonnet Myers Theorem and other comparison theorems, we must ensure that the connection on a finite dimensional Riemannian manifld (M,g) is geodesically complete. We can determine whether (M,g) is geodesically complete using the following criteria:

(i) If and only if its metric space is complete (Hopf-Rinow)

(ii) Its exponential map $\exp_p$ can be defined on the entire TM for all p $\in$ M and therefore γ can be extended between $(-\infty, \infty)$

(iii) Any two points, say p, q, can be connected by a length minimising geodesic whose distance d(p,q) is the minimum length of all possible curves in the direction p to q.

(iv) All bounded closed subset in M are compact.

Proof. We now prove (i) by showing that a Cauchy sequence, $(a_n)_{n\in\mathbb{N}}$ contains a converging subsequence a_{nj} and is therefore self-converging.

(i): Let $\gamma_X : j \to M$ be a normal geodesic with a derivative $\dot{\gamma}(0) = X \in T_pM$. We initially assume that sup j := $t^+ < \infty$, which we will later show to be a contradiction. We then use the following re-parametrisation,

$$len\,\gamma_X|_{[}s_i, s_j] = |s_i - s_j|$$

and apply it on a_n, obtaining

$$d(\gamma_X(a_n), \gamma_X(a_m)) = len\,\gamma_X|_{[}a_n, a_m] = |a_n - a_m|$$

where a_n- a_m. This shows that the Cauchy sequence$((\gamma_X)a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence within (M, dist) whose limit $\lim_{n\to\infty} \gamma_X(a_n)$ converges to a point p$\in$ M. Similarly, for another sequence b_n in j, $\gamma(b_n)$ also converges to p since

$$d(\gamma_X(a_n), \gamma_X(b_n)) = len\,\gamma_X|_{[}a_n, b_n] = |a_n - b_n|$$

. Hence the geodesic can be extended.
(ii): If E= TM, then (M,g) must be geodesically complete
(ii) $\Rightarrow$ (iii): We omit the proof here since it is too complicated.
(iv) $\Rightarrow$ (i): This is straightforward to prove as it is simply a standard assumption in topology. □

6.1 Riemannian Distance and the Length Minimising Property

We are more concerned about piecewise smooth curves than simply smooth parametric curves when dealing with the Riemannian distance.

Definition 6.3. We begin defining the Riemannian distance between two points p,q by setting a piecewise smooth curve $\gamma\colon [a,b] \to M$. The tangent vector T$_{\gamma t}M$ is given by $\dot{\gamma}(t) = d\gamma\frac{d}{dt}$ for all t$\in$ $[a,b]$. The length of the curve, L(γ), can be written as

$$L(\gamma) = \int_a^b \sqrt{< \dot{\gamma}(t), \dot{\gamma}(t) >_{\gamma(t)}}\, dt$$

. If L(γ) = d(p,q), we call the curve length minimising.

Proposition 6.4. L(γ) is invariant when transformed by any other parametrizations, i.e the arc length of γ

$$\lim_{m(K)\to 0} l(k) = \int_a^b ||\dot{\gamma}(t)||$$

(which we refer to as the limit here) remains the same after re-parametrization.

Proof. (Proof 1) For every $\epsilon > 0$ there exists a $\delta > 0$ such that the equation

$$||\gamma(t^{'}) - \gamma(t)|| - |t^{'} - t| \cdot ||\gamma(t))|| < \epsilon|t^{'} - t|$$

holds true for all t$_{i-1}$, t $\in$ [a,b] where $|t_{i-1} - t_i| < \delta$. If we make the assumption that this also holds true for every element in the sub-interval [t$_{i-1}$, t$_i$] for i=1, ...N. This yields

$$|l(k) - \Sigma_{i=1}^{N}||\dot{\gamma}(t_{i-1})|| \cdot |t_i - t_(i-1)| = |\Sigma_{i=1}^{N}||\gamma(t_i) - \gamma(t_{i-1})|| - ||\dot{\gamma}(t_i)|| \cdot (t_i - t_(i-1))| \tag{4}$$

$$< \Sigma_{i=1}^{N}\epsilon(t_i - t_(i-1)) \tag{5}$$

We can now see that as m(K)$\to$0, equation (10) approaches $\int_a^b ||\dot{\gamma}(t)||$. If we apply the chain rule, the limit now tends to

$$\int_a^b ||\dot{\gamma}(h(t))|| \cdot \frac{dh}{dt}(t)|dt = \int_a^b ||\dot{\gamma}^{'}(h(t))|| \cdot \frac{dh}{dt}dt \tag{6}$$

$$= \int_a^b ||\dot{\gamma}^{'}(t')||dt^{'} \tag{7}$$

$$= L(\gamma^{'}) \tag{8}$$

We have shown that the arc length bounded between the closed interval [a,b] $\subseteq$ ℝ is always equal to ϵ(b-a). The length of the curve is therefore insensitive to any forms of re-parametrizations. □

Proposition 6.5. All geodesics are length minimising. However, this property is only unique up until re-parametrisation.

7 Curvature

7.1 Riemannian Curvature Tensor

From this section onwards, we assume that all manifolds are Riemannian manifolds.

Definition 7.1. Since the Riemannian Curvature Tensor is C$^\infty(M)$ linear in X,Y,Z $\in \mathfrak{X}$ for any point p$\in$ M, it can be defined by a collection of trilinear maps R$_p$: $T_pM^3 \to T_pM$. The Riemannian Curvature Tensor is given by

$$R(X,Y)Z = \nabla_Y(\nabla_X Z) - \nabla_Y(\nabla_Y Z) - \nabla_{[Y,X]Z}$$

Definition 7.2. Properties of the Riemannian Curvature Tensor:

(i) The covariant derivatives do not commute

$$R(X,Y)Z = \nabla^2 Z(Y,X) - \nabla^2 Z(X,Y)$$

(ii) Only the Levi-Civita connection is required to define the curvature of a manifold $R(X,Y)Z\Big|_p$ only depends on X$\Big|_p$, Y$\Big|_p$ and Z$\Big|_p$.

Proposition 7.1. The Riemannian Curvature Tensor is a (1,3) tensor field.

7.1.1 Defining the Bianchi Identities

Proposition 7.2. The Bianchi Identities are extremely important when exploring the geometrical properties of a Riemannian manifold and the behavior of the Riemann curvature tensor under coordinate transformations.

The First Binachi Identity states that the cyclic sum of the derivatives of the Christoffel symbols and that of the products of Christoffel symbols with themselves are equal to zero. This implies that the Riemann curvature tensor is antisymmetric in the last three indices. By the definition of the Riemann curvature tensor,

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}$$

, we can introduce the cyclic permuatation of the indices i,j, k in each term

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

We can re-write this using local coordinates

$$\partial_i \Gamma_{jkl} - \partial_j \Gamma_{ikl} + \Gamma_{jmn}\Gamma_{imk} - \Gamma_{imn}\Gamma_{jkl} + \partial_k \Gamma_{ijl} - \partial_i \Gamma_{kjl} + \Gamma_{kmn}\Gamma_{ijn} - \Gamma_{ikm}\Gamma_{jln} \\ + \partial_j \Gamma_{ikl} - \partial_k \Gamma_{ijl} + \Gamma_{jmn}\Gamma_{ikn} - \Gamma_{ijn}\Gamma_{kmn} = 0$$

$$\Rightarrow \partial_I \Gamma_{jkl} + \partial_j \Gamma_{kIl} + \partial_k \Gamma_{Ijl} + \Gamma_{jmn}\Gamma_{Imk} + \Gamma_{kmn}\Gamma_{Ijn} + \Gamma_{Ikn}\Gamma_{jmn} = 0$$

The Second Bianchi Identity shows that the sum of some second-order derivatives of the Christoffel symbols and some covariant derivatives of the Christoffel symbols are equal to zero.It demonstrates the cyclic symmetry of the Riemann curvature tensor under permutation of the last three indices. Using the covariant derivative from the First Bianchi Identity,

$$\nabla_i (\partial_j \Gamma_{kil} + \partial_k \Gamma_{ijl} + \Gamma_{jmn}\Gamma_{imk} + \Gamma_{kmn}\Gamma_{ijn} + \Gamma_{ikn}\Gamma_{jmn}) = 0$$

, we can apply the Leibniz rule which allows us to obtain

$$\partial_i \partial_j \Gamma_{kil} + \nabla_i \partial_j \Gamma_{kil} + \partial_i \partial_k \Gamma_{ijl} + \nabla i \partial_k \Gamma_{ijl} + \partial_i \Gamma_{jmn}\Gamma_{imk} + \nabla_i (\Gamma_{jmn}\Gamma_{imk}) + \\ \partial_i \Gamma_{kmn}\Gamma_{ijn} + \nabla_i (\Gamma_{kmn}\Gamma_{ijn}) + \partial_i \Gamma_{ikn}\Gamma_{jmn} + \nabla_i (\Gamma_{ikn}\Gamma_{jmn}) = 0.$$

Since Christoffel symbols are symmetric as shown in lemma , we can re-write the above as

$$\partial_i \partial_j \Gamma_{kil} + \partial_j \partial_i \Gamma_{kil} + \partial_i \partial_k \Gamma_{ijl} + \partial_k \partial_i \Gamma_{ijl} + \partial_i \Gamma_{jmn}\Gamma_{imk} + \Gamma_{jmn}\partial_i \Gamma_{imk} + \\ \partial_i \Gamma_{kmn}\Gamma_{ijn} + \Gamma_{kmn}\partial_i \Gamma_{ijn} + \partial_i \Gamma_{ikn}\Gamma_{jmn} + \Gamma_{ikn}\partial_i \Gamma_{jmn} = 0$$

$$\Rightarrow \partial_i \partial_j \Gamma_{kil} + \partial_i \partial_k \Gamma_{ijl} + \partial_i \Gamma_{jmn}\Gamma_{imk} + \Gamma_{jmn}\partial_i \Gamma_{imk} + \partial_i \Gamma_{kmn}\Gamma_{ijn} + \Gamma_{kmn}\partial_i \Gamma_{ijn} \\ + \partial_i \Gamma_{ikn}\Gamma_{jmn} + \Gamma_{ikn}\partial_i \Gamma_{jmn} = -(\partial_j \partial_i \Gamma_{kil} + \partial_k \partial_i \Gamma_{ijl} + \partial_i \Gamma_{ikn}\Gamma_{jmn} \\ + \Gamma_{ikn}\partial_i \Gamma_{jmn} + \partial_i imk + \Gamma_{jmn}\partial_i \Gamma_{imk} + \partial_i \Gamma_{kmn}\Gamma_{ijn} + \Gamma_{kmn}\partial_i \Gamma_{ijn})$$

Definition 7.3. (Number of individual components in the Riemannian tensor)We understand that the following symmetry conditions for R$_{ijkl}$from the Bianchi identities.

(i) Since R_{ijkl} is anti-symmetric within each of the two groups ij and kl, the number of independent conditions is given by N=$\binom{n}{2}$

(ii) Subsequently, R_{ijkl} is symmetric when swapping ij with kl

$$R_{klij} = R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

. We can therefore refine the number of independent conditions to $\frac{N(N+1)}{2}$ or simply $(\binom{n}{2} + 1)$

(iii) If all the indices are different for a manifold with diemension n$\geq$ 4, there are a total of ^4C_n possibilities since $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ from proposition 7.2.

As a result, the total number of individual components is given by

$$\frac{1}{2}\binom{n}{2}(\binom{n}{2} + 1) - \binom{n}{4} = \frac{n^2(n^2-1)}{12}$$

7.2 Sectional Curvature

Definition 7.4. For a point p$\in$ M and a 2-dimensional subspace or plane $\Pi \subset T_pM$, we define the sectional curvature of Π at p as a real number where

$$\begin{aligned} K_p(\Pi) &= K_p(x, y) \\ &:= \frac{Rm_p(x, y, y, x)}{g_p(x, x)g_p(y, y) - (g_p(x, y))^2} \end{aligned}$$

where x,y are vectors $\in \Pi$ which form the basis and Rm denotes the Riemann curvature tensor.

Definition 7.5. We define the span of manifold, span(M), as the maximum number of linearly independent vector fields $\mathfrak{X}(M)$ that it can admit.

Remark 7.3. Properties of the sectional curvature:

(i) K(x,y) is independent of the choice of x, y $\in \Pi$

(ii) If the vector basis {x,y} is orthonormal, then $K(\Pi) = R_m(x, y, y, x)$

(iii) If two vectors u,v $\in T_pM$ are linearly independent, then this equation holds

$$K(u, v) := K(span\{u, v\})$$

Proof. We prove (i) by setting $(\tilde{x}, \tilde{y})$ as a set of vectors which form a new basis of Π. We can relate the 2 basis by the equations

$$\tilde{x} = ax + by$$
$$\tilde{y} = cx + dy$$

where ad-bc $\neq 0$.

$$< \tilde{x}, \tilde{x} >< \tilde{y}, \tilde{y} > - < \tilde{x}, \tilde{y} >^2 = (ad - bc)^2 < x, x >>< y, y > \\ - < x, y >$$

Using (ii) from Remark 7.3, we can write $\text{Rm}(\tilde{x}, \tilde{x}, \tilde{y}, x_2)$ by factoring twice repeatedly as

$$\begin{aligned} &Rm(ax + by, cx + dy, cx + dy, ax + by) \\ &= adRm(\tilde{x}, \tilde{y}, c\tilde{x} + d\tilde{y}, a\tilde{x} + by_2) \\ &+ bcRm(\tilde{y}, \tilde{x}, c\tilde{x} + d\tilde{y}, a\tilde{x} + b\tilde{y}) \\ &= ad(adRm(\tilde{x}, \tilde{y}, \tilde{y}, \tilde{x}) + bcRm(\tilde{x}, \tilde{y}, \tilde{x}, \tilde{y})) \\ &+ bc(adRm(\tilde{y}, \tilde{x}, \tilde{y}, \tilde{x}) + bcRm(\tilde{y}, \tilde{x}, \tilde{x}, \tilde{y})) \\ &= (ad - bc)^2 Rm(x, y, y, x) \end{aligned}$$

which corresponds to equation (21). This therefore shows that K(x,y) is independent to the choice of basis.

Subsequently, we can prove (iii) by □

Proposition 7.4. The Riemannian curvature tensor is determined by the sectional curvature.

Proof. □

Corollary 7.5. *Let i,j,k,l be variables in* $\{1, \dots n\}$ *and let* $\{e^k\}$ *be an orthonormal basis of* T_pM. R_{ijkl} *can be written as*

$$R_{ijkl} :=< R(e_i, e_j)e_k, el >$$

. If $R_{ijkl} = K_0(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})$ *where* δ *denotes the Kronecker delta, then the sectional curvature is constant and* $K_0 = 1$.

Lemma 7.6. *For a Riemannian manifld (M,g) and a positive constant* λ *where* $\tilde{g} = \lambda g$*, we can define K as the sectional curvature of K and similarly* $\tilde{K}$ *for* $\tilde{g}$*. Hence, the sectional curvature of K and* $\tilde{K}$ *are related by*

$$\tilde{K} = \frac{1}{\lambda} K$$

.

Proof. Let ϕ be a locally defined map $(x^1 \dots x^n)$ around a point p$\in$ M between the two metrics. By the definition of the Christoffel symbols in Def 5.5, since $\Gamma^k_{ij} = \tilde{\Gamma^k_{ij}}$, the coefficients of the tensor field for both metrics g and $\tilde{g}$ are equivalent $\mathrm{R}_{ijkl} = \tilde{R_{ijkl}}$ (hence the tensors). Additionally, for two linearly independent vectors, u,v $\in T_pM$,

$$\begin{aligned} K_p(\tilde{x}, \tilde{y}) \colon &= \frac{Rm_p(x,y,y,x)}{\tilde{g}_p(x,x)\tilde{g}_p(y,y) - (\tilde{g}_p(x,y))^2} \\ &= \frac{\lambda g_p(R(x,y)x,y)}{\lambda g_p(x,x)\lambda g_p(y,y) - \lambda^2 g_p(x,y)^2} \\ &= \frac{\lambda g_p(R(x,y)x,y)}{\lambda^2(g_p(x,x)g(y,y) - g_p(x,y)^2)} \end{aligned}$$

□

7.3 Ricci Curvature

Definition 7.6. The Ricci curvature tensor is a (0,2) tensor field and is a contraction of the Riemannian curvature tensor.

Definition 7.7. Setting x=z_n as a unit vector and hence $\{z_i\}_1^{n-1}$ as the orthonormal basis to the hyperplane of T_pM which is orthogonal to the span of x, the Ricci curvature is defined by

$$Ric_p(x) = \frac{1}{n-1}\sum_{i=1}^{n-1} < R(x,z_i)x, z_i >$$

The Ricci scalar (scalar curvature),the simplest invariant of the curvature of a Riemannian manifold, is given by

$$\begin{aligned} S(p) &= \frac{1}{n}\sum_{j=1}^{n-1} Ric_p z_j \\ &= \frac{1}{n(n-1)}\sum_{i,j=0}^{n-1} < R(z_i,z_j)z_i, z_j > \end{aligned}$$

Remark 7.7. Properties of the Ricci curvature:

(i) The Ricci curvature can be expressed locally as

$$Ric = g^{kl}R_{kijl}dx^i \otimes dx^j$$

where g denotes the metric of the manifold.

(ii) When tracing over any two arguments, the result is either K_0 iff R_{ijij}= -R_{ijji} or 0 in all other cases

(iii) The Ricci curvature tensor is also symmetric by the Bianchi identities

(iv) If $\{e_j\}$ is an orthonormal basis, then

$$Ric(x, y) = \sum_{i=1}^{n} Rm(e_j, x, y, e_j)$$

Proposition 7.8. The Ricci curvature is independent to the choice of orthonormal basis.

Proof. For x,y $\in T_pM$, we define a trace of the Ricci curvatre Q(x,y) as a bilinear map of the function z $\mapsto$ R(x,z)y. As Q corresponds to a self-adjoining map K, Q(x,y) can be expressed as $< K(x), y >$. By setting an orthonormal basis x=$\{z_i\}_1^{n-1}$,

$$tr(K) = \sum_{j=1}^{n-1} < K(z_j), z_j >$$
$$= \sum_{j=1}^{n-1} Q < z_j, z_j >$$

By Definition 7.7, the trace can be written as

$$(n-1) \sum_{j=1}^{n-1} Ric_p(zj)$$

$$= K_p[n(n-1)]$$

, which shows that the Ricci curvature of a manifold is scalar invariant. □

8 Bonnet Myers Theorem

Proposition 8.1. (Second Variation of Arc Length) Every vector field V along a smooth curve α splits into 2 components, one being parallel to $\alpha^{'}$ and the other being orthogonal to $\alpha^{'}$. We shall focus on the orthogonal component, nor V, where nor V is defined by

$$norV = V - a^{'} < V, a^{'} >$$

. For a geodesic γ: $[0, a] \to M$, we define $L_x^{''}(0)$ as the second variation of arc length

$$L_x^{''}(0) = \frac{1}{c} \int_0^a < nor\, V^{'}, nor\, V^{'} > - < R(V, \gamma^{'}), V, \gamma^{'} > dt$$

where c is a positive constant and x is the variation of γ along V.

Theorem 8.2. *(Bonnet Myers Theorem) The Bonnet Myers theorem states that for any complete Riemannian manifold (M^n,g) whose sectional curvature, $K \geq \delta$, where δ is a positive constant, its Ricci curvature, R, satisfies:*

$$Ric(M) \geq \delta(n-1)$$

. We can then estimate its diameter, diam(M), since it is always bounded by

$$sup_{p,q \in M}\, dist(p, q) \leq \frac{\sqrt{\pi}}{\delta}$$

.

Proof. We begin by assuming a contradiction that d(p,q) > $\frac{\sqrt{\pi}}{\delta}$. The Hopf-Rinow theorem states that there exists a unique minimising geodesic γ between two points p,q that is defined between [0,L] since the metric S is complete. We can vary γ by a unit vector of w_0 where $w_0 \cdot \gamma_0^{'} = 0$. Hence, for a parallel transport w(s) of w_0, $w_s \cdot \gamma_s^{'} = 0$.

We now establish a vector field V(s) and consider its first and second derivatives

$$V(s) = w(s) sin(\frac{\pi t}{L})$$
$$V^{'}(s) = w(s) \frac{\pi}{L} cos(\frac{\pi s}{L})$$
$$V^{''}(s) = -w(s)(\frac{\pi}{L})^2 sin(\frac{\pi s}{L})$$

By the second variation of arc length,

$$\begin{aligned} L_x^{''}(0) &= -\int_0^L < V'' - R\gamma^{'}(V, \gamma^{'}), V > \\ &= -\int_0^L < -w(\frac{\pi}{L})^2 sin(\frac{\pi s}{L}) - R(sin\frac{\pi s}{L})(w, \gamma^{'})\gamma^{'}, w(sin\frac{\pi s}{L}) > \\ &= \int_0^L (sin^2 \frac{\pi s}{L})((\frac{\pi}{L})^2 - < w, w > - < R(w, \gamma^{'})W, \gamma^{'} >) \\ &= \int_0^L (sin^2 \frac{\pi s}{L})((\frac{\pi}{L})^2 - K(w, \gamma^{'}))dt \end{aligned}$$

Since $\delta = K(w, \gamma')$, the above expression must be $\leq \int_0^L (sin^2 \frac{\pi s}{L})((\frac{\pi}{L})^2 - \delta)dt$.

We have now shown that if the value of s is infinitesimally small, then $\mathrm{L}_x(s) L_x(0) = \mathrm{L}_\gamma$. As the curve $\mathrm{L}_x(s)$ has a shorter length than $\gamma(p, q)$, it is therefore a contradiction. Thus, since the Bonnet Myers theorems shows that (M^n,g) is closed and bounded by a subset of M, (M^n,g) must be compact. □

Corollary 8.3. *The covering space argument states that all compact manifolds with Ric(M) > 0 have a finite fundamental group $\pi_1(M)$.*

Proof. Let $\tilde{M} \to M$ be a universal cover of M and that we can apply a pullback metric from M to $\tilde{M}$. Then the sectional curvature $\tilde{K_(}p)$ also satisfies $\tilde{K} \geq \delta$ and is also compact. From Definition 2.10, there exists a bijection between the fibre $\pi^{-1}(p)$ and every element of the fundamental group $\pi_1(M)$. As $\pi^{-1}(p)$ is discrete, it is finite and therefore $\pi_1(M)$ is finite. □

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References

[Cheeger et al., 1975] Cheeger, J., Ebin, D. G., and Ebin, D. G. (1975). *Comparison theorems in Riemannian geometry*, volume 9. North-Holland publishing company Amsterdam.

[Do Carmo and Flaherty Francis, 1992] Do Carmo, M. P. and Flaherty Francis, J. (1992). *Riemannian geometry*, volume 6. Springer.

https://www.ma.imperial.ac.uk/ mtaylor5/rgnotes.pdf
https://www.cis.upenn.edu/ cis6100/cis61008Riem-conn.pdf