Schramm-Loewner Evolution

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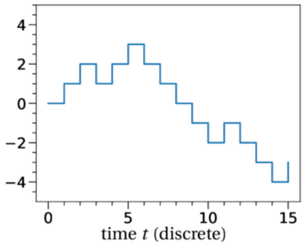
Brownian Motion

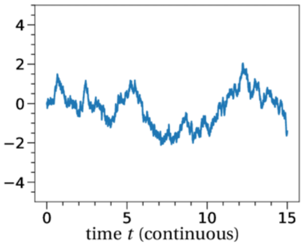
For the basis of this project we considered random walks and stochastic processes. Brownian motion is an example of a stochastic process that appears in real world. A stochastic process B_t is called a one-dimensional Brownian motion if

1. $B_0 = 0$.
2. B_t has independent increments.
3. B_t is a.s continuous.
4. $B_{t+s} - B_s \stackrel{\mathscr{D}}{=} \mathscr{N}(0, t)$.

Brownian Motion Continued

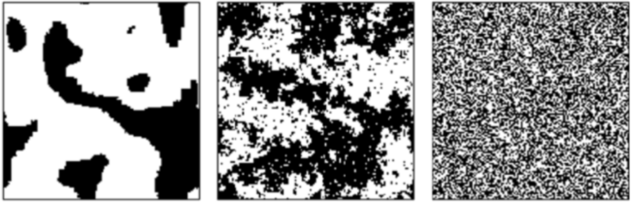
It is important for us to note that Brownian motion is the scaling limit of a simple random walk. Also conformal maps i.e angle preserving maps composed with Brownian motion are also a Brownian motion.





Motivatons

The motivation began the Schramm-Loewner Evolutions came from lattice models in physics. Stochastic models were being studied at the start of the 20th Century, and it was theorized that the limiting distributions were conformally invariant, and shared universal properties. The main motivator was the Ising model, which involves ferromagnetism.



Schramm-Loewner evolution

Definition of Schramm Loewner Evolution

Let $\kappa > 0$. A Schramm-Loewner Evolution, SLE(κ) is the random collection of conformal maps g_t that come from solving the Loewner Ordinary-Differential Equation

$$\dot{g}_t = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0 = 0, z \in \mathbb{H} \tag{0.1}$$

where B_t is a Brownian motion.

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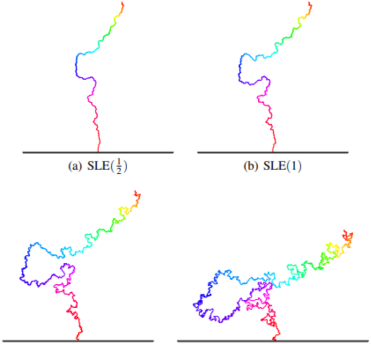
Schramm's Principle

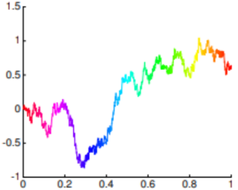
Schramm-Loewner evolutions are the only random curves satisfying conformal invariance and domain Markov property.

Schramm-Loewner Visualised



(a) SLE($\frac{1}{2}$) (b) SLE(1)

(c) SLE(2) (d) SLE(4)



(e) The standard Brownian motion sample driving the SLE's.

Percolation Model

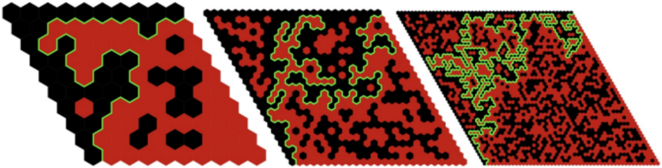
We will be considering the site percolation model. Consider a coffee filter, with some areas being closed and others being open. When coffee drips and meets an area that is closed, it will move left or right (randomly) to the next 'hole'.

Site Percolation

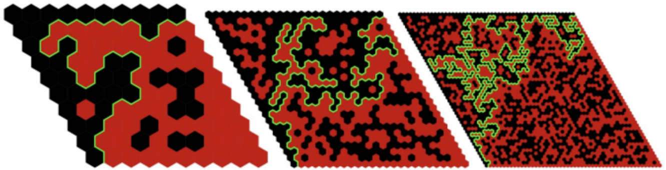
An $n \times n \times n$ lattice where each site is open with probability p and closed with probability $1 - p$.

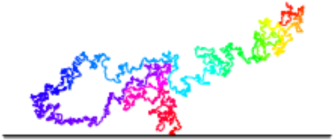
The task is, for a given p, what is the probability that a path exists from top to bottom.

Percolation Visualised



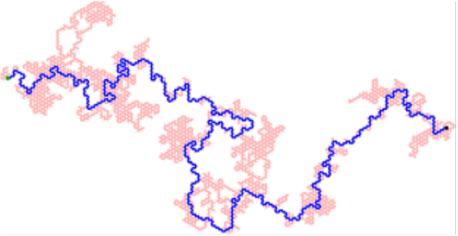
Percolation Visualised



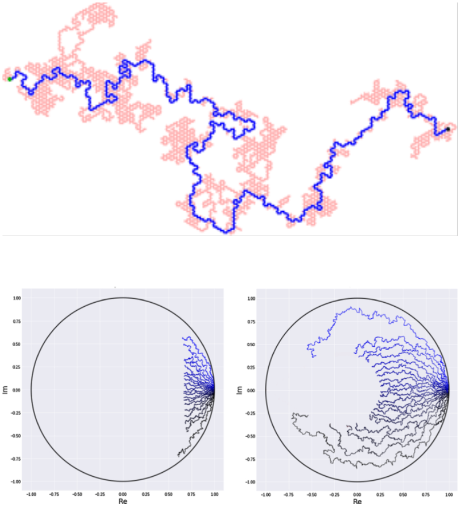


(a) SLE(6)

Loop-Erased Random Walk



Loop-Erased Random Walk



Further Applications

Schramm-Loewner Evolutions have been applied to

1. Turbulent systems and the study of Two-Dimensional, incompressible Navier-Stokes.
2. Help define certain types of Quantum Field Theory.