On Schramm-Loewner Evolutions

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Abstract

In this paper we review topics in stochastic calculus and complex analysis to provide a basis for the Loewner equation. We present properties of $\mathbb{H}$-hulls and their capacities and demonstrate their importance to the Loewner equation. We characterize Brownian motion as a driving term for the Loewner equation, and show that Schramm-Loewner evolutions are the random curves that satisfy the Loewner equation. Finally, we discuss properties of Schramm-Loewner evolutions and connect these random curves to stochastic processes, such as the percolation model, in physics and show their importance as scaling limits.

1. Introduction

Schramm-Loewner evolutions became a tool for modeling stochastic systems recently. Over the last century, stochastic processes were an emerging topic that helped describe many processes in the real world such as viruses, stock markets, particle physics, percolation, and so on. The challenge and also convenience of these systems was that they are driven by a random motion; however, the almost universal constant was that the process was almost always Brownian motion. Studies in complex analysis and hydronamic relations by Charles Loewner led him to establish the Loewner Differential Equation

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \lambda(t)}.$$

Oded Schramm discovered that with $\lambda(t)$ being a Brownian motion, the set of solutions to the differential equation were a random collection of conformal maps. These random curves were also the only curves that satisfied conformal invariance and the domain Markov property. These curves, Schramm-Loewner evolutions, began to appear in many models in physics and other random processes. The results were that the scaling limit of a percolation model converged (in distribution) to SLE(6), and that certain loop-erased random walks in distribution were equal to radial SLE(2). In recent years, Schramm-Loewner evolutions began appearing in random processes and even helped to show that the fractal dimension of standard Brownian motion is $\frac{4}{3}$. The interesting part of Schramm-Loewner evolutions was that seperate physical models and stochastic processes could be related with SLEs which have us only deal with Brownian motion.

2. Stochastic Processes and Equations

In this chapter, we focus on *stochastic calculus* and Brownian Motion.

2.1. **Brownian Motion.**

Traditionally, probability was typically considered through a combinatorial lens. However, people began to consider it analytically. Modern day probability theory is credit to Andrew Nikolaevich Kolmogorov when he combined sample spaces, and measure theory to create an axiom system for probability theory in 1933. Random walks started to gain interest in the early 20th century due to real world applications, primarily biology. Alongside, many processes were created - Markov process, Wiener Process, poisson process, etc. We will be using $(\Omega, \mathscr{F}, \mathsf{P})$ as our probability space, where Ω is the measurable space, $\mathscr{F}$ is a σ-algebra, and P is the probability measure on $\mathscr{F}$. Note that a σ-algebra on a set S is a non-empty collection of subsets of S closed under complements, countable unions, and countable intersections.

Definition 2.1. A *stochastic process* is a collection of random variables X_t indexed by time which belongs to an ordered set I. We denote $(X_t)_{t\in I}$ as a stochastic process.

Remark. Typically, $I = \mathbb{R}_{\geq 0}$, or $I = \mathbb{Z}_{\geq 0}$, and we call these cases continuous 'time stochastic process' and 'discrete time stochastic process', respectively.

We call the mapping $t \mapsto X_t(\omega)$ the path of $(X_t)_{t\in I}$. One of the most famous and crucial processes with a continuous path is Brownian motion.

Definition 2.2. A stochastic process $(B_t)_{t\geq 0}$ is called a (1D) Brownian motion if

(1) $B_0 = 0$.
(2) $B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}, \ldots, B_{t_n} - B_{s_n}$ are independent for any $n \in \mathbb{N}$ and for any $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$.
(3) For any $s, t \geq 0$, $B_{s+t} - B_s$ is normally distributed with mean 0 and variance t.
(4) With probability one, $t \to B_t$ is continuous.

Remark. We will present a stronger definition later in the chapter.

Remark. Note that X is normally distributed with mean μ and variance σ^2 when $\mathbb{P}[X \in A] = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$ For any Borel subset $A \in \mathbb{R}$.

The original theory of Brownian motion came from physical experiments on particle motion in alcohol. Robert Brown is credit with the discovery of Brownian motion in 1827 for his work in botany, yet other scientists have observed this irregularity earlier. The interest in Brownian motion has increased over time due its consistent appearance in mathematics, both pure and applied, economics, and statistical physics. We often come across random walks, and what is interesting is that Brownian motion is the scaling limit of simple random walks. Another point of interest of Brownian motion is its fractal behavior. Though in statistics, Brownian motion, on a macro scale, may resemble a 'nice' function, it is a fractal in all small neighborhoods. Further note, Brownian motion is described by the Wiener process, a continuous-time stochastic process.

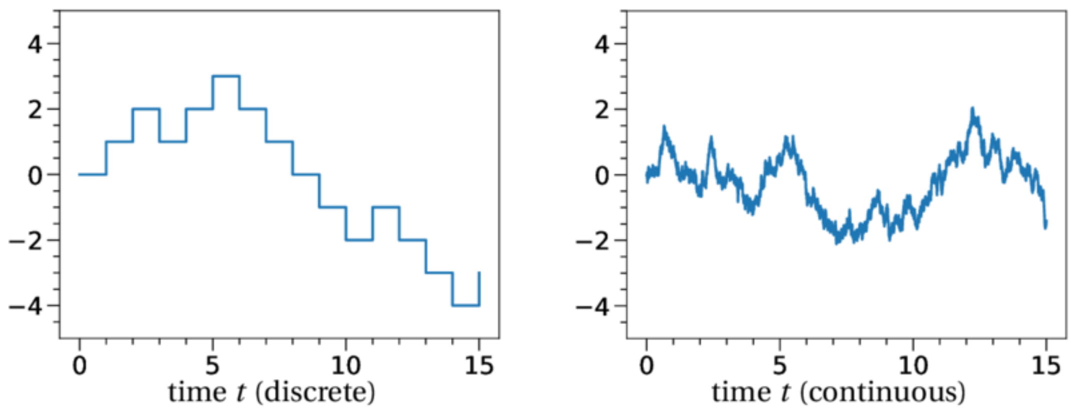


FIGURE 1. Random walk and Brownian motion. On the left, we have a random walk defined on $\mathbb{Z}$ and changes at discrete times T. On the right we have a standard Brownian motion, starting at 0 and takes values in $\mathbb{R}_{t\geq 0}$.[CT20]

Lemma 2.1. $dB_t \sim \sqrt{dt}$

Proof. $dB_t := B_{t+dt} - B_t \sim \mathcal{N}(0, dt) = \sqrt{dt}$ □

Definition 2.3. A *filtration* on $(\Omega, \mathscr{F})$ is a sequence of σ-algebras, $(\mathscr{F}_t)_{t\geq 0} \subset \mathscr{F}$ such that for any $0 \leq t_1 \leq t_2, \mathscr{F}_{t_1} \subset \mathscr{F}_{t_2}$

A stochastic process X_t on $(\Omega, \mathscr{F})$ is adapted to the filtration $\mathscr{F}_t$ if for each $t \geq 0, X_t$ is $\mathscr{F}_t$-measurable This allows to create a stronger definition of Brownian motion.

Definition 2.4. A process $(B_t)_{t\geq 0}$ is called a (standard one-dimensional) Brownian motion with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if $B_0 = 0$ and

(1) $B_t - B_s$ are independent from $\mathscr{F}_s$ for any $0 \leq s < t$.
(2) $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.
(3) With probability one, $t \mapsto B_t$ is continuous.

Remark. We can consider statement (1) as the present and future is independent of the past.

Define the quadratic variation of a process $(X_t)_{t\geq 0}$ as

$$V_X^2(t) = \lim_{\max_k(t_{k+1}-t_k)} \sum_{k=0}^{p} |X_{t_{k+1}} - X_{t_k}|^2 \tag{2.1}$$

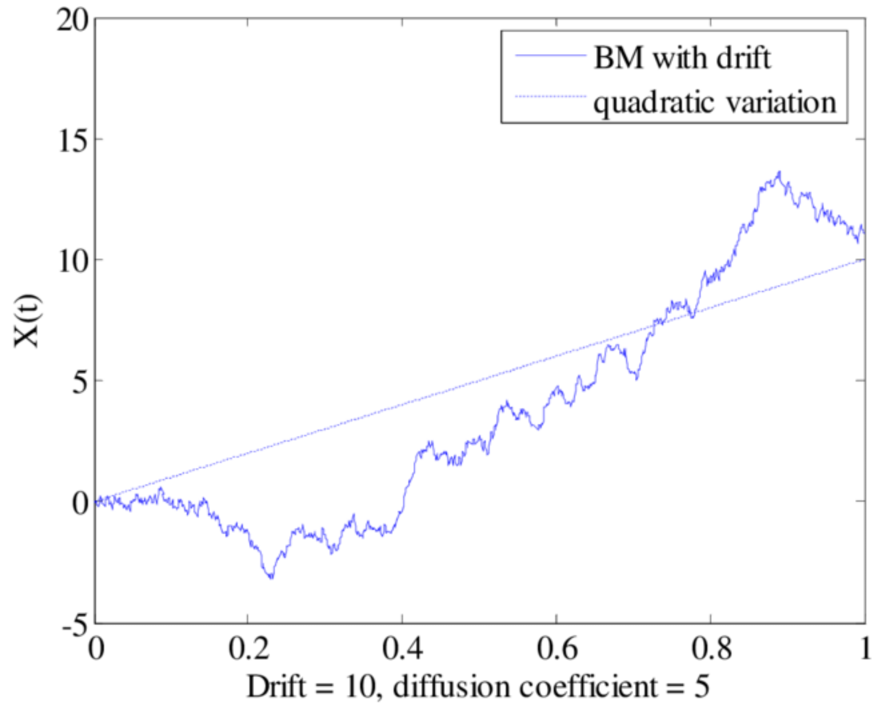


FIGURE 2. Sample path of Brownian motion with drift. [Kas11]

And in general, the kth variation as

$$V_X^k(t) = \lim_{\max_k(t_{k+1}-t_k)} \sum_{k=0}^{p} |X_{t_{k+1}} - X_{t_k}|^k \tag{2.2}$$

where $0 = t_0 < t_1 < \ldots < t_p = t$.

2.2. **Stochastic Integration.**

Stochastic calculus, also called Itô Calculus, is important for study of Brownian motion. We can define the stochastic process Y_t as an integral.

$$Y_t(\omega) = \int_0^t f(t,\omega)dB_t(\omega) \tag{2.3}$$

As a note to the reader, stochastic integration does not behave as a Reimann integral because the total variation of Brownian motion is infinite. We will present an example later in the chapter. For stochastic integration, we need to consider the correct set of f. For now, they will be measurable, adapted, and square-integrable processes. Let us call the subset of L^2, $\mathscr{L}^2$, that for $T > 0$ is the set of measurable processes f that satisfy

$$\mathbb{E}[\int_0^T f(t,\omega)^2 dt] < \infty \tag{2.4}$$

If f can be expressed as

$$f(t,\omega) = \sum_{k=0}^{n-1} X_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t) \tag{2.5}$$

where $0 \leq t_0 < t_2 < \ldots < t_n \leq T$. Then we call that $f \in \mathscr{L}^2$ simple.

Let us also define the mapping $f \mapsto I[f]$ such that $I[\mathbb{1}_{[s,t)}] = B_t - B_s$. Therefore

$$I[f] = \sum_{k=0}^{n-1} X_k(B_{t_{k+1}} - B_{t_k}) \tag{2.6}$$

Lemma 2.2. *For any bounded, simple $f \in \mathscr{L}^2$*

$$\mathbb{E}[I[f]^2] = \mathbb{E}[\int_0^T f(t,\omega)^2 dt] \tag{2.7}$$

Proof. On the right hand side,

$$\mathbb{E}[\int_0^T f(t,\omega)^2 dt] = \sum_{k=0}^{n-1} \mathbb{E}[X_k^2(t_{k+1} - t_k)] \tag{2.8}$$

On the left hand side,

$$\mathbb{E}[If]^2] = \sum_k \mathbb{E}[X_k^2(B_{t_{k+1}} - B_{t_k})^2] + 2\sum_{k<l} \mathbb{E}[X_k X_l(B_{t_{k+1}} - B_{t_k})(B_{t_l+1} - B_{t_l})] \tag{2.9}$$

Now if we compare

$$\mathbb{E}[X_k^2(B_{t_{k+1}} - B_{t_k})^2] = \mathbb{E}[X_k^2]\mathbb{E}[(B_{t_{k+1}} - B_{t_k})^2] = \mathbb{E}[X_k^2](t_{k+1} - t_k). \tag{2.10}$$

$$\mathbb{E}[X_k X_l(B_{t_{k+1}} - B_{t_k})(B_{t_1+1} - B_{t_1})] = \mathbb{E}[X_k X_l(B_{t_{k+1}} - B_{t_k})\mathbb{E}[B_{t_l+1} - B_{t_l}] = 0. \tag{2.11}$$

□

Definition 2.5. For any $f \in \mathscr{L}^2$ the stochastic integral (Often called the Itô integral) is defined as

$$\int_0^T f dB_t(\omega) := I[f](\omega). \tag{2.12}$$

Lemma 2.3. *For any $f \in \mathscr{L}^2, \mathbb{E}[(\int_0^T f dB_t)^2] = \mathbb{E}[\int_0^T f^2 dt]$*

Proof. $\mathbb{E}[(\int_0^T f^2(dB_t)^2] = \mathbb{E}[(\int_0^T f^2(\sqrt{dt})^2] = \mathbb{E}[\int_0^T f^2 dt]$ □

An example of stochastic integration, consider the following integral;

$$\int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2} \tag{2.13}$$

Proof. At first we might expect the integral to be $B_t^2/2$, however

$$\mathbb{E}[\int_0^t (B_s - f_n(s,\omega))^2 ds] = \mathbb{E}[\sum_j \int_{t_j}^{t_j+1} (B_s - B_{t_j})^2 ds] = \frac{1}{2}\sum_j (t_{j+1} - t_j)^2 \to 0. \tag{2.14}$$

$$\sum_j B_j(B_{t_{j+1}} - B_{t_j} = \frac{B_t^2}{2} - \frac{1}{2}\sum_j (B_{t_{j+1}} - B_{t_j})^2. \tag{2.15}$$

The latter sum converges in L^2 to t. □

Another useful tool in stochastic calculus is Itô's formula. Using itô's formula, functions of Brownian motion can be expressed as a sum of a stochastic integral and an integral with respect to dt.

Theorem 2.4. *(Itô's formula) Let $F : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function such that $\dot{F}, F', F''$ exist and are continuous, where*

$$\dot{F}(t,x) = \frac{\partial F}{\partial t}(t,x), F'(t,x) = \frac{\partial F}{\partial x}(t,x), and F'' = \frac{\partial^2 F}{\partial x^2}(t,x). \tag{2.16}$$

Then almost surely,

$$F(t,B_t) = F(0,B_0) + \int_0^t \dot{F}(s,B_s)ds + \int_0^t F'(s,B_s)dB_s + \frac{1}{2}\int_0^t F''(s,B_s)dB_s \tag{2.17}$$

for any $t \in \mathbb{R}_+$.

Remark. The proof of this is based on the Taylor expansion of $F(t,x)$.

The second order terms help explain the result in the example above.

2.3. Stochastic Differential Equations.

If $(X_t)_{t\in[0,T]}$ be a continuous stochastic process and let $(B_t)_{t\in\mathbb{R}_{\geq 0}}$. Then we say X_t satisfies the stochastic differential equation

$$dX_t = \alpha(t,X_t)dt + \beta(t,X_t)dB_t \tag{2.18}$$

where $X_0 = A$ if for each $t \in [0,T]$

$$X_t = A + \int_0^t \alpha(s,X_s)ds + \int_0^t \beta(s,X_s)dB_s \tag{2.19}$$

For more material on Probability Theory the reader can use Durret's book on probability theory.

3. Loewner Equation

In this chapter we will cover topics in Complex Analysis and the Loewner Differential Equation.

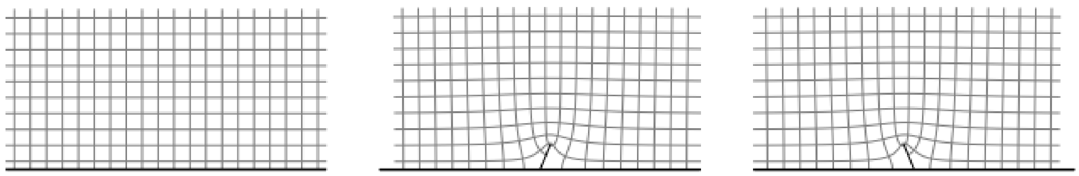
3.1. Conformal Maps.

Let $\mathbb{D}$ represent the unit disc i.e $\mathbb{D} = \{z \in \mathbb{C}||z| < 1\}$.
Let $\mathbb{H}$ be the upper half of the complex plane i.e $\mathbb{H} = \{z \in \mathbb{C}|Im(z) > 0\}$. We denote $\partial\mathbb{S}$ as the boundary of $\mathbb{S}$.

Definition 3.1. If f is a differentiable function and $U \subset \mathbb{C}$, we call a map $f : U \mapsto \mathbb{C}$ conformal if it preserves angles for all points in U.

Remark. It is easier to think of this as the angle between the tangents is preserved, and to prove this we simply consider the scalar product.

This is equivalent to saying that the function is analytic and one-to-one. We are interested in conformal maps because they involve analytic functions. The intersection between calculus and conformal maps motivates our usage of them. The divergence, or net flux, are similar to harmonic functions. Calculating the divergence over a body with internal boundaries is often simplified by considering a conformal map to $\mathbb{H}$. This is also the motivation for many hydro-based definitions.



(a) The grid lines in $\mathbb{H}$.

(b) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{R}}(z)$.

(c) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{L}}(z)$.

FIGURE 3. Conformal maps from $\mathbb{H}$ onto the complements of line-segments.[Kem17]

Theorem 3.1. *(Reimann mapping theorem) Let $V \subset \mathbb{C}$ be a simply connected domain with $v \in V$. Then there exists a unique conformal map f that maps V onto $\mathbb{D}$ such that $f(v) = 0$ and $f'(v) > 0$*

Proof. We can assume $f'(z_0)$ is positive and real. Our goal is to show that f is surjective. So let's assume that there exists a $\nu \in \mathbb{D}$ such taht $f(z) = \nu$ has no solution in V. Then consider a $g \in f_n$ with $|g'(z_0)| > |f'(z_0)$, which would lead to a contradiction for our choice of f. So now consider $\psi_\nu : \mathbb{D} \mapsto \mathbb{D}$ where

$$\psi_\nu(z) = \frac{\nu - z}{1 - \overline{\nu} z} \tag{3.1}$$

and let

$$G(z) = \sqrt{\psi_\nu \circ f(z)}. \tag{3.2}$$

Note that $G(z)$ is zero free, since $\psi_\nu(z) = 0$ if and only if $z = \nu$. Thus we can define a holomorphic branch of $\log \psi_\nu \circ f(z)$, and we can choose a holomorphic branch for $G(z)$ with

$$G(z) = e^{\frac{1}{2}\log \psi_\nu \circ f(z)}. \tag{3.3}$$

Thus we can $g(z)$ as

$$g(z) = \psi_{G(z_0)} \circ G(z). \tag{3.4}$$

Note that g is injective. Now our goal is to show $|g'(z_0)| > |f'(z_0)|$. So note that

$$f(z) = \psi_\nu^{-1} \circ s \circ \psi_{G(z_0)}^{-1} \circ g(z) = \phi \circ g(z), \tag{3.5}$$

where $s(x) = x^2$. Now if we compute $\phi(0)$, we get that $\phi(0) = f(0) = 0$. Now note that if $f : \mathbb{D} \mapsto \mathbb{D}$ is a holomorphic map where $f(0) = 0$, then

$$|f(z)| < |z|, \quad \forall z \in \mathbb{D}. \tag{3.6}$$

So by 3.6 $|\phi(z)| < |z|$, thus $|\phi'(0)| \leq 1$. Since ϕ is injective, then $|\phi'(0)| < 1$. Then

$$f'(z_0) = \phi'(g(z_0)) \cdot g'(z_0) = \phi'(0) \cdot g'(z_0). \tag{3.7}$$

Hence $|f'(z_0)| < |g'(z_0)|$. □

Theorem 3.2. *Let $U \in \mathbb{C}$ be a bounded domain. A conformal map $f : \mathbb{D} \mapsto U$ extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ if and only if ∂U is locally connected.*

We will now present an important definition that we will use throughout the chapter.

Definition 3.2. A set $K \in \bar{\mathbb{H}}$ is called a hull if K is compact and $\mathbb{H}\backslash K$ is simply connected.

From this we arrive at this interesting theorem.

Theorem 3.3. *For any hull K, there exists a unique conformal map $f_K : \mathbb{H}\backslash K \mapsto \mathbb{H}$ such that*

$$\lim_{z\to\infty} (f_K(z) - z) = 0 \tag{3.8}$$

Such f_K is said to have hydrodynamic normalization. Near ∞, f_K has the expansion

$$f_K(z) = z + \sum_{n\geq 1} a_n z^{-n} \ \mid \ a_n \in \mathbb{R}. \tag{3.9}$$

Proof. Let $\Gamma : \mathbb{H}\backslash K \mapsto \mathbb{D}$ be a conformal map, then by the holomorphic extension of $z \mapsto \Gamma(-1/z)$ to a neighborhood of 0, then $\Gamma(\infty) \in \partial\mathbb{D}$ is well-defined using the Schwarz reflection principle. Using 3.1 there are conformal maps from $\mathbb{H}\backslash K$ onto $\mathbb{H}$ which map ∞ to ∞. Call one of these maps ϕ. Denote Φ as

$$\Phi(z) = \frac{-1}{\phi(-1/z)}. \tag{3.10}$$

By the Schwarz reflection principle, f extends holomorphically to a neighborhood of 0. Now consider $\varepsilon > 0$ where $B(0,\varepsilon) \cap \mathbb{H} \subset \{-1/z \mid \}$, then f maps $B(0,\varepsilon) \cap \mathbb{H}$ into $\mathbb{H}$. Thus

$$\Phi(z) = b_1 z + b_2 z^2 + \cdots \tag{3.11}$$

near 0 where $b_1 > 0, b_i \in \mathbb{R}$. Then for large $|z|$

$$\phi(z) = c_1 z + c_2 + c_3 z^{-1} + c_4 z^{-2} + \cdots \tag{3.12}$$

where, similarly, $c_1 > 0, c_i \in \mathbb{R}$. So if $c_1 = 1$ and $c_0 = 0$ we are done. Now note that if $f : \mathbb{H}\backslash K \mapsto \mathbb{H}$ is a conformal onto map taking ∞ to ∞, then all other maps can be expressed as a composition of a Mobius self-map of $\mathbb{H}$ fixing ∞ and f. So there is a unique choice of a Mobius self-map such that f_K has the expansion

$$f_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots \tag{3.13}$$

for $z \in \mathbb{H}\backslash B(0, R)$. $\square$

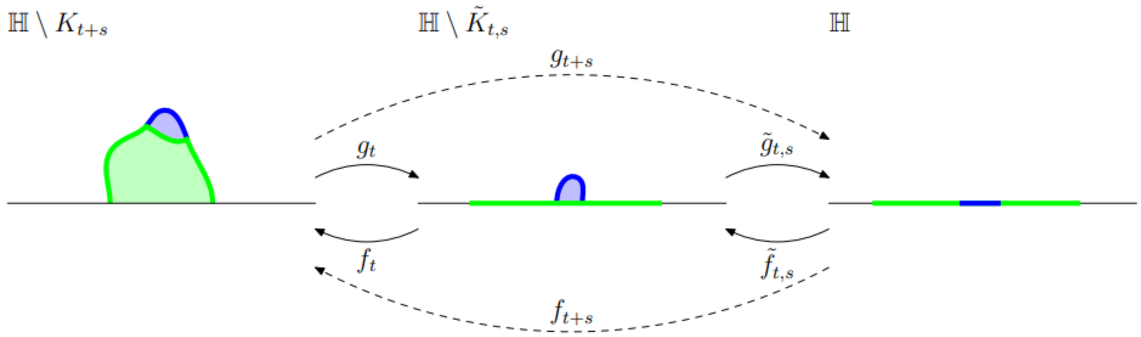


Figure 4. Composition of hydro-dynamical maps.[Kem17]

3.2. **Loewner Equation for Simple Curves.**

Definition 3.3. For a hull K and f_K that satisfies the hydrodynamic normalization, then the coefficient a_1, call $a_1(K)$, in the expansion of f_k is called the half-plane capacity of K.

Note that the half-plane capacity satisfies:

$$a_1(cK) = c^2 a_1(K) \tag{3.14}$$

$$a_1(K \cup L) = a_1(K) + a_1(f_K(L)) \tag{3.15}$$

$$a_1(K + x) = a_1(K) \tag{3.16}$$

Let $\gamma : [0, \infty) \mapsto \bar{\mathbb{H}}$ be a simple curve and $\gamma(0) = 0$ be a simple curve where $\gamma(0, \infty) \subset \mathbb{H}$, then the family of hulls $K_t = \gamma[0, t]$.

Definition 3.4. A family of hulls K_t is said to be parameterized with half-plane capacity if $a_1(K_t) = 2t$.

Now consider $K_t = \gamma[0, t]$ with $\mathbb{H}_t = \mathbb{H}\backslash K_t$ and let $g_t : \mathbb{H}_t \mapsto \mathbb{H}$ be the corresponding maps. Let $W(t) = g_t(\gamma(t))$. Then g_t satisfies the Loewner differential equation in the upper half-plane.

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \quad g_0 z = z. \tag{3.17}$$

In 1923, Charles Loewner was studying cofmral maps from the unit-disc, and introduced the Loewner equation in $\mathbb{D}$ where

$$\partial_t f_t(z) = f_t'(z) z \frac{z + e^{iU_t}}{z - e^{iU_t}}. \tag{3.18}$$

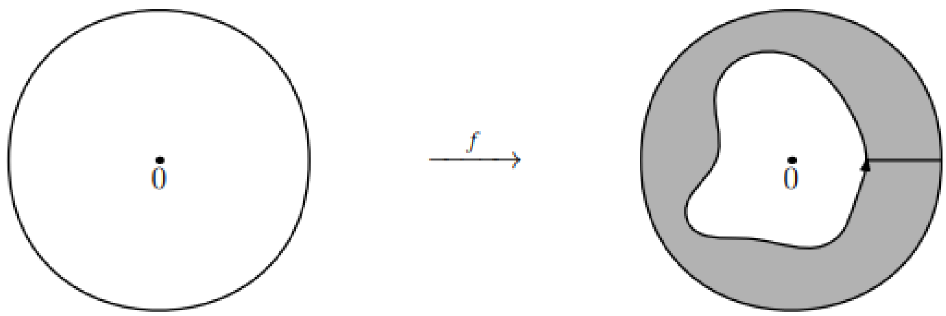


FIGURE 5. A map f from $\mathbb{D}$ into $\mathbb{D}$ can be studied by the Loewner equation in $\mathbb{D}$ by defining a curve that first goes from $\partial\mathbb{D}$ to $\partial f(\mathbb{D})$.[Kem17]

Theorem 3.4. *Let $T > 0$ and let $\gamma : [0, T] \mapsto \mathbb{C}$ be a simple curve such that $\gamma(0) = 0$ and $\gamma(0, T) \subset \mathbb{H}$. If γ is parameterized by the capacity, then*

$$W(t) = \lim_{z \to \gamma(t)} g_t(z) \tag{3.19}$$

exists for any $t \in [0, T]$ and $t \mapsto W(t)$ is continuous. Furthermore the hydro-dynamically normalized conformal maps g_t related to γ satisfy the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{3.20}$$

with initial value $g_0(z) = z$.

Proof. First we will show that there exists a constant C such that if $K \subset B(x_0, r) \cap \mathbb{H}$ and $|z| > Cr$ then

$$|g_k(z) - z - \frac{a_1(K)}{z}| \leq \frac{Cra_1(K)}{|z|^2}. \tag{3.21}$$

Let $f_K = g_K^{-1}$, then near ∞

$$f_K = z - a_1 z^{-1} + \cdots \tag{3.22}$$

Let $h(z) = \Im(f_K(z) - z)$. Note that h is harmonic in $\mathbb{H}$. So we can rewrite h using the Poisson kernel as

$$h(z) = \Im \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta - z} h(\zeta) d\zeta. \tag{3.23}$$

Since $h = \Im(f_K)$ on $\mathbb{R}$, so we can express $f_K(z)$ as

$$f_K(z) = z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\zeta - z}\Im(f_K(\zeta))d\zeta. \tag{3.24}$$

Let I be the smallest interval containing $\zeta \in \mathbb{R}|f_K(\zeta) \in \mathbb{H} \cup K$. Then $f_K(\zeta) = 0$ outside of I. Therefore, for large enough z,

$$f_K(\zeta) = z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\zeta - z}\Im(f_K(\zeta))d\zeta = z - \sum_{n\geq 1}(\int_I \zeta^{n-1}\Im(f_K(\zeta))d\zeta)z^{-n}. \tag{3.25}$$

Thus,

$$|g_K(z) - z - \frac{a_1(K)}{z}| = |\frac{1}{\pi}\int_I (\frac{1}{\zeta - z} + \frac{1}{z})\Im(f_K(\zeta))d\zeta| \leq a_1(K)\sup(|\frac{1}{x-z} + \frac{1}{z}| \mid x \in I). \tag{3.26}$$

Since $I \subset (-3r, 3r)$, then

$$|\frac{x}{(x-z)z}| \leq 6r|z|^2 \tag{3.27}$$

for any $|z| \geq 6r, x \in I$. Q.E.D

We also need to show that if $\Phi : [0,\infty) \mapsto \mathbb{R}^n$ is a function whose right derivative exists for all t and the map $t \mapsto \Phi'_+(t)$ is continuous, then Φ is continuously differentiable and $\Phi'(t) = \Phi'_+(t)$.

We can assume that $\Phi(0) = 0$ and $\Phi'_+(t) = 0$.Then let $w = \infty(t \mid |\Phi(t)| > \epsilon t)$ where $\epsilon > 0$. We will now go for a contradiction. So assume that $p < \infty$. Since $\Phi(w) \in C^1$ then Φ is constant. Also, since $\Phi'_+(t) = 0$, there exists a $\delta > 0$ such that $f(w+x) < \epsilon w + \epsilon x = \epsilon(w+x), \forall\, 0 < x < \delta$. Contradiction, since w is the infimum. Therefore, $w = \infty$ and Φ is differentiable. Q.E.D

For a given $t > 0$ consider $\gamma(t, t+\varepsilon)$ and the image S under g_t. The curve starts at $W(t)$. Then by (2.8) $g_{t+\varepsilon} = g_S \circ g_t$. If we apply (2.14) to g_S and let $g_t(z) = v$:

$$|g_S(v) - v - \frac{a_1(S)}{v - W(t)}| \leq \frac{Cra_1(S)}{|z - W(t)|^2}. \tag{3.28}$$

Therefore,

$$|g_{t+\varepsilon}(z) - g_t(z) - \frac{a_1(S)}{g_t(z) - W(t)}| \leq \frac{Cra_1(S)}{|g_t(z) - W(t)|^2}. \tag{3.29}$$

Now if we divide by ε and take the limit $\varepsilon \to 0$ we get:

$$\lim_{\varepsilon\to 0}\frac{g_{t+\varepsilon}(z) - g_t(z)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{a_1(S)/\varepsilon}{g_t(z) - W(t)} + \lim_{\varepsilon\to 0} C\frac{ra_1(S)/\varepsilon}{|g_t - W(t)|^2}. \tag{3.30}$$

Since we have $a_1(\gamma((0,t])) = 2t$, then $\lim_{\varepsilon\to 0}\frac{a_1(S)}{\varepsilon} = 2$. Thus,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \quad 0 \leq t < T_z. \tag{3.31}$$

□

4. Schramm-Loewner Evolution

4.1. Schramm-Loewner Evolution.

Schramm-Loewner evoltuions were discovered by Oded Schramm in 1999. Originally, they were called stochastic Loewner evolutions. His paper revealed that random curves can be described using the Loewner equation with a random driving term. These evolutions were motivated by theoretical physics, especially the Ising model.

Definition 4.1. A Loewner chain is the solution g_t of the Loewner differential equation with a continuous driving term.

Definition 4.2. Let $\kappa \geq 0$. A chordal Schramm-Loewner evolution SLE(κ) is a stochastic Loewner chain with a driving process W_t equal to a Brownian motion with variance parameter κ. That is, SLE(κ) is the random collection of conformal maps g_t that come from solving the Loewner Ordinary-Differential Equation

$$\dot{g_t} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0 = 0, z \in \mathbb{H}. \tag{4.1}$$

Schramm's Principle: Schramm-Loewner evolutions are the only random curves satisfying conformal invariance and the domain Markov property.
Let's say we have a collection of probability measures $\pi^{(U,a,b)}$ denoted by all triples (U, a, b) such that U is a simply connected domain and a, b are two boundary points of U. Say that $\pi^{(U,a,b)}$ is the law of a random curve $\gamma : [0, \infty)$ such that $\gamma([0, \infty)) \subset \overline{U}$ and $\gamma(0) = a, \gamma(\infty) = b$. Then we say the family $\pi^{(U,a,b)}$ satisfies the domain Markov property if for all (U, a, b) and for every $t \in \mathbb{R}_{\geq 0}$ and for any measurable set B in the space of curves

$$\pi^{(U,a,b)}(\gamma|_{[t,\infty)} \in B|\mathscr{F}_t) = \pi^{(U\setminus\gamma([0,t]),\gamma(t),b)}(\gamma \in B) \tag{4.2}$$

where $\mathscr{F}_t$ is the filtration generated by $\gamma(t)$.

Theorem 4.1. *Let K_t be SLE(κ) and let $W(t)$ be the corresponding Brownian motion with respect to a filtration $\mathscr{F}_t$. Then SLE(κ) satisfies*

1. *Scaling: For any $c > 0, cK_{t/c^2} \stackrel{D}{=} K_t$.*
2. *Conformal Markov Property: For any $s \in \mathbb{R}_{\geq 0}$ the family of hulls*

$$\hat{K_{t,s}} = \overline{g_s(K_{s+t}\setminus K_s) - W_s} \tag{4.3}$$

is independent of $\mathscr{F}_s$ and $\hat{K_{s,t}} \stackrel{D}{=} K_t$.

Proof. The hulls and the conrresponding conformal maps are

$$\begin{aligned} \text{Scale invariance: } & cK_{t/c^2}, cg_{t/c^2}(z/c) \\ \text{Conformal invariance: } & \overline{g_s(K_{s+t}\setminus K_s) - W - s}, \hat{g}_{s,t}(z). \end{aligned} \tag{4.4}$$

of scale invariance and the conformal Markov property respectively, where $\hat{g}_{s,t}(z) = g_{s+t} \circ g_s^{-1}(z + W_s) - W_s$. If we differentuate wuth respect to t, then we notice that these chains satisfy the Loewner equation with driving processes cW_{t/c^2} and $W_{s+t} - W_s$ respectively. □

Definition 4.3. Let K_t be a SLE(κ) and let D be a simply connected domain with a and b, $a \neq b$ being two boundary points of D. We define SLE(κ) in domain D going from a to b to be the image of K_t under any conformal map $\phi : \mathbb{H} \mapsto D$ such that $\phi(0) = a, \phi(\infty) = b$.

Definition 4.4. For any Loewner chain g_t, we define the generating curve γ as

$$\gamma(t) = \lim_{\varepsilon \downarrow 0} g_t^{-1}(W(t) + i\varepsilon). \tag{4.5}$$

The function γ is called the trace of the Loewner chain.

Theorem 4.2. *For each κ, the trace γ exists and is a random curve such that the hulls K_t of SLE(κ) are generated by γ.*

Proof. First we claim that if $t \mapsto F_t(y)$ converges to some γ uniformly on compact subsets of $[0, T)$ as $y > 0$ tends to 0, then γ is a continuous curve and K_t is generated by γ. Moreover, for each $T \in [0, t)$ the map $z \mapsto f_t(z)$ extends continuously to $\overline{\mathbb{H}}$. The proof of this statement can be found in [source]. Thus it is enough for us to show that the functions $t \mapsto f_t(W_t + iy)$ converges uniformly as $y \to 0$.
So for each $\kappa \neq 8, \exists$ a constant $\tau > 0$, and a random variable X which is finite with probability 1 such that

$$|f'_t(i2^{-n})| \leq X 2^{n(1-\tau)} \tag{4.6}$$

for all $t \in \mathscr{D}_{2n}, n \in \mathbb{N}$. Note that W_t is a Brownian motion, then there exists a finite, with probability 1, random variable χ such that

$$|W_{t+s} - W_t| \leq \chi \sqrt{s \log(1/s)} \tag{4.7}$$

. for any $t, s \in [0, 1]$.
Let $t \in [0, 1], y \in (0, 1)$. Take $n \in \mathbb{N}$ and $t_0 \in \mathscr{D}_{2n}$ such that

$$2^{-n} \leq y < 2^{-n+1}, \quad t_0 \leq t < t_0 + 2^{-2n}. \tag{4.8}$$

Now we claim also that there exists a constant A such that for any solution f_t of the Loewner equation for the inverse Loewner map and for any $x + yi \in \mathbb{H}, t \in \mathbb{R}_{\geq 0}$ and $s \in [0, y^2]$

$$A^{-1}|f_t'(x+iy)| \leq |f_{t+s}'(x+iy)| \leq A|f_t'(x+iy)|$$

$$|f_{t+s}(x+iy) - f_t(x+iy)| \leq Ay|f_t'(x+iy)|. \tag{4.9}$$

We show this by differentiating the Loewner equation. Using the fact that $|x+iy-W_t| \geq y$ and the triangle inequality we get that

$$|\partial_t f_t'(x+iy)| \leq \frac{2|f_t''(x+iy)|}{y} + \frac{2|f_t'(x+iy)|}{y^2}. \tag{4.10}$$

To approximate $|f_t''(z)|$ for a fixed z, consider $\psi(\zeta) = x + iy\frac{1-\zeta}{1+\zeta}$. Therefore, ψ is a Mobius map from $\mathbb{D}$ onto $\mathbb{H}$ and it has the expansion

$$\psi(\zeta) = x + iy(1 + 2\sum_{n\geq 1}(-1)^n\zeta^n). \tag{4.11}$$

It follows that the function $(f_t \circ \psi(\zeta) - f_t(z))/(f_t'(z)\psi'(0))$ has the expansion

$$\zeta + \frac{f_t''(z)(\psi'(0))^2 + f_t'(z)\psi''(0)}{2f_t'(z)\psi'(0)}\zeta^2 + \cdots \tag{4.12}$$

around $\zeta = 0$. It follows that $|f_t''(z)||\phi'(0)|^2 \leq |f_t'(z)|(|\psi''(0)| + 4|\psi'(0)|)$ and so $|f_t''(z)| \leq 6|f_t'9z)|y^{-1}$. So if we combine the previous estimates we arrive at

$$-\frac{14}{y^2} \leq \partial_t \log|f_t'(x+it)| \leq \frac{14}{y^2} \tag{4.13}$$

where $-|z| \leq \Re(z) \leq |z|$.The proof is finished by integrating the inequality with respect to t, and the latter claim comes from substituting the former into the Loewner equation, and then integrated with respect to t. Q.E.D. From all above it follows that

$$\begin{aligned}|f_t'(W_t+iy)| \leq \lambda|f_{t_0}'(W_t+iy)| \leq \lambda|f_{t_0}'(W_t+iy_0)| \leq \lambda(1+\frac{|W_t-W_{t_0}|^2}{y_0^2})^3|f_{t_0}'(W_{t_0}+iy)| \\ \leq \lambda n^r 2^{n(1-\tau)} \leq y^{\tau-1}\rho(1/y) \quad (4.14)\end{aligned}$$

for some subpower function ρ. Note that λ is a generic constant. So now let's integrate our current bound. Then for any $0 < y_1 < y_2 \leq y < 1$, by the triangle inequality we get

$$\begin{aligned}|f_t(W_t+iy_2) - f_t(W_t+iy_1)| &\leq \int_{y_1}^{y_2} |f_t'(W_t+iu)|du \\ &\leq \int_0^y u^{\tau-1}\rho(1/u)du = y^{tau}\rho(1/y)\end{aligned} \tag{4.15}$$

where $\rho(x) = \int_0^1 u^{\tau-1}\rho(x/u)du$. Thus $\gamma(t) = \lim_{y\to 0} f_t((W_t+iy)$ exists and satisfies

$$|\gamma(t) - f_t(W_t+iy)| \leq y^\tau\rho(1/y). \tag{4.16}$$

Therefore γ is continuous and generate K_t. $\square$

Theorem 4.3. *Let the random curve $\gamma : [0,\infty) \mapsto \overline{\mathbb{H}}$ be SLE(κ). Then*

(1) *For all $0 < \kappa \leq 4$, γ is simple.*

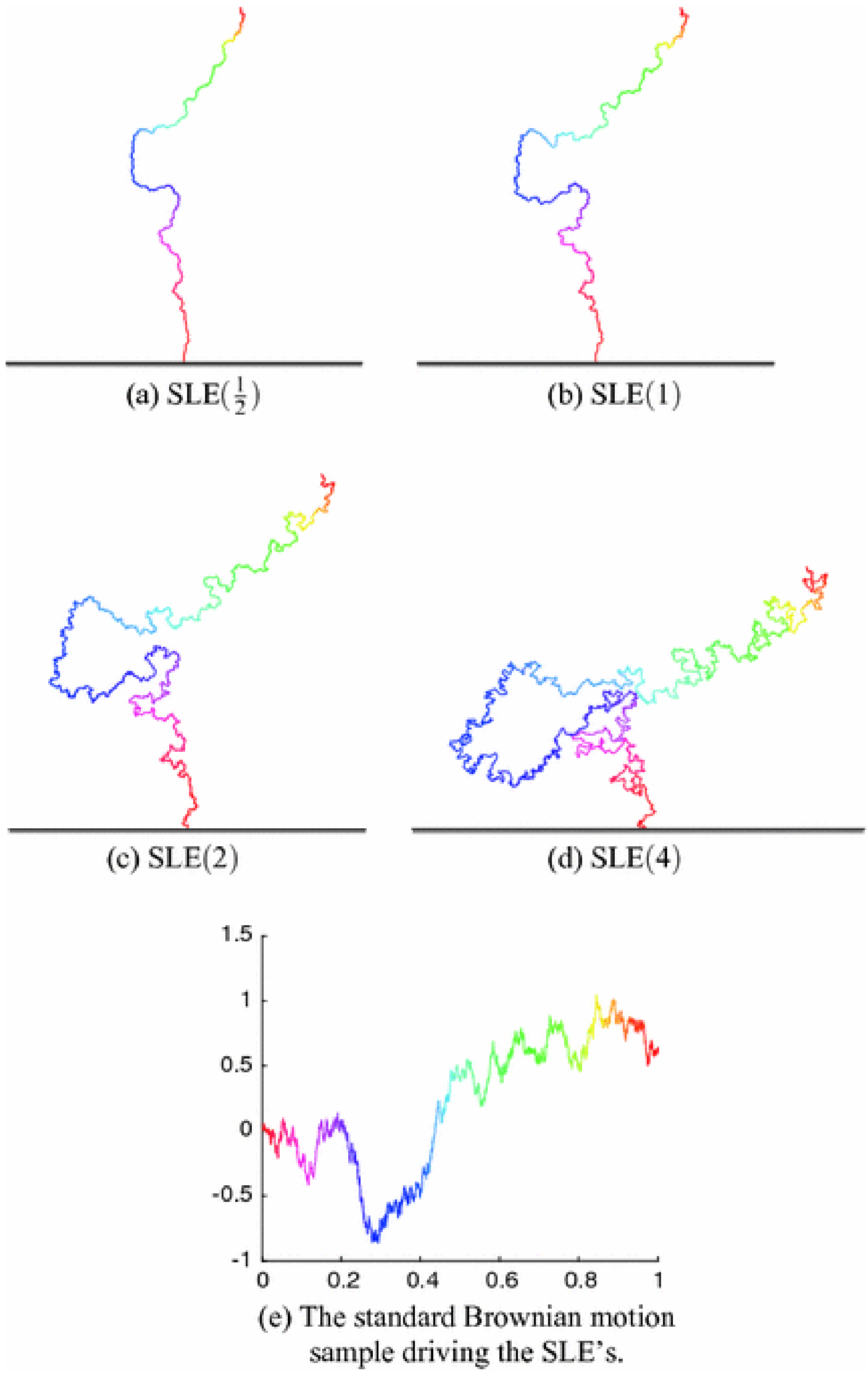


FIGURE 6. SLE(κ) for $\kappa = \frac{1}{2}, 1, 2, 4$ i.e the standard Brownian motion multiplied by a $\sqrt{\kappa}$.[Kem17]

(2) *For all* $4 < \kappa < 8, \gamma$ *is not simple on any interval.*
(3) *For all* $\kappa \geq 8, \gamma$ *is not simple and is space filling.*

Proof. For the proof please see [RS04] $\square$

4.2. **Calculations.**
As mentioned earlier, Schramm-Loewner evolutions were motivated by their use in physics. The random curves appear in specific models were there are barriers that separate near-quantum level properties. A famous lattice model in statistical physics is the Ising model. Each vertex v is occupied by an elementary magnet, spin, which takes values $\sigma_v \in \{\pm 1\}$. The model is defined

$$H(\underline{\sigma}) = -\Sigma \sigma_v \sigma_w \tag{4.17}$$

where $\underline{\sigma} = (\sigma_v)_{v \in V}$ is the spin configuration of the system and V is a finite subset of the the square lattice $\mathbb{Z}^2$. What makes Schramm-Loewner evolutions relevant is that they are the

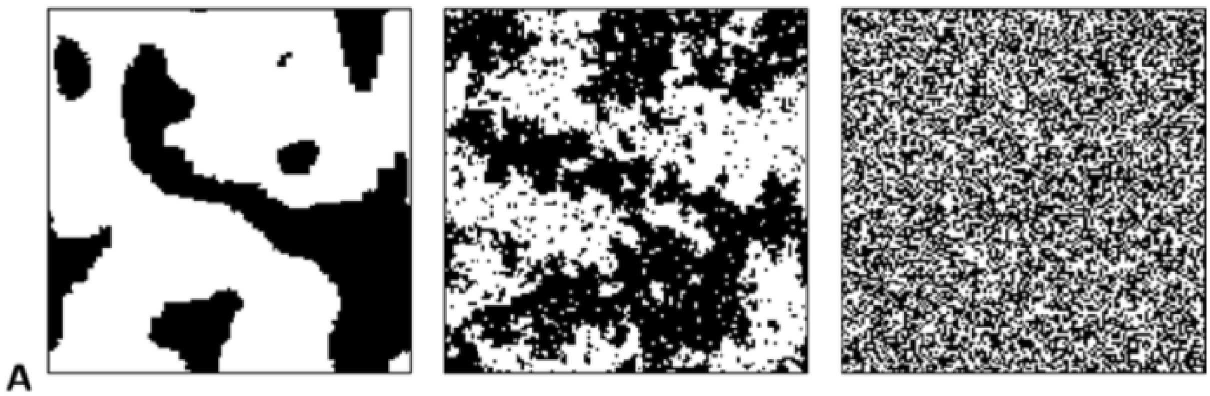


FIGURE 7. Ising model simulations of a dynamic system at critical and non-critical temperatures.[KSCB09]

universal scaling-limit of many models in physics.
First we will be considering the site percolation model. Consider a coffee filter, with some areas being closed and others being open. When coffee drips and meets an area that is closed, it will move left or right (randomly) to the next 'hole'. If that 'hole' is again closed, it will again move, and so on until there is an open hole and coffee drips through. Now consider $G = (V, E)$ be a finite or infinite graph. We will consider G as a lattice or a sub-graph of a lattice. Let $p \in [0, 1]$ be a parameter and consider a family of random variables and assign one to each vertex of G taking values in open, closed. We say that a vertex $v \in V$ is open or closed depending on the value of the random variable. We assume that

$$\mathbb{P}[v = open] = p, \quad \mathbb{P}[v = closed] = 1 - p \tag{4.18}$$

for each $v \in V$.
Now consider a rhombus, or in general k-length rhombi $R_k = \{x + y \exp i\pi/3 \mid x, y \in [1, k]\}$. Let $f(p, k)$ denote the probability of a left to right crossing in R_k. It is important to notice that f is monotone. Furthermore $\lim_{k\to\infty} f(p, k) = \{0, \frac{1}{2}, 1\}$ when $p < \frac{1}{2}, p = \frac{1}{2}, p > \frac{1}{2}$,

respectively. It is also important to note that if we chose a different aspect ratio for $p = \frac{1}{2}$ the limit of the crossing probability is not necessarily $\frac{1}{2}$.

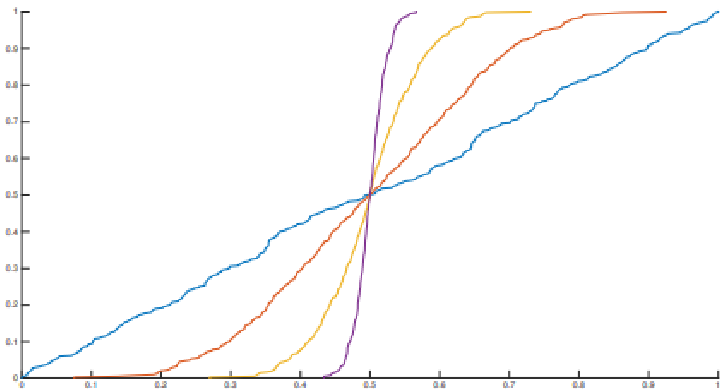


FIGURE 8. Crossing probabilities of a left to right crossing in a rhombus of side length k. The graph is a function of p for different values of k ($k = 1$ blue, $k = 4$ orange, $k = 16$ yellow,$k = 64$ purple). Sample size is 200 for each k.[Kem17]

Let's consider a hexagonal lattice in the upper half-plane. Now consider this model, and at each step choose randomly, e.g flipping a coin, decide to choose turning right or turning left, e.g heads means go right and tails means go left. Or another way of thinking about this is at each step color the square green or red. Let's say this lattice has $mesh\delta$ and let $\delta \to 0$. Then the interface between the green and red squares converges, in distribution, to SLE_6 - this was proved by Stanislav Smirnov. Here are some interesting theorems on crossing probabilities for different shapes.

Theorem 4.4. *(FKG inequality) For increasing non-negative random variables X, Y in a percolation model, it holds that* $\mathbb{E}[XY] \geq \mathbb{E}[X]\mathbb{E}[Y]$

Corollary 3.5. (Crossing probability for long rhombi or rectangles) For any $\rho \geq 1$ there exists $\varepsilon \in (0,1)$ such that for every $n \in \mathbb{Z}_{\geq 0}$

$$\varepsilon \leq \mathbb{P}_{p_c}[\mathscr{S}_{L-R}(R(v, \lceil \rho n \rceil, n))] \leq 1 - \varepsilon. \tag{4.19}$$

Where $\mathscr{S}_{L-R}$ is the event of a left-right crossing of a rhombus R.

Corollary 3.6. (Crossing of an annuli) There exists a constant c_1 and $\Delta_1 > 0$ such that for any $z_0 \in \mathbb{C}$ and $1 < r < R$

$$\mathbb{P}_{p_c}[\mathscr{S}(A(z_0, r, R))] \leq c_1(r/R)^{\Delta_1} \tag{4.20}$$

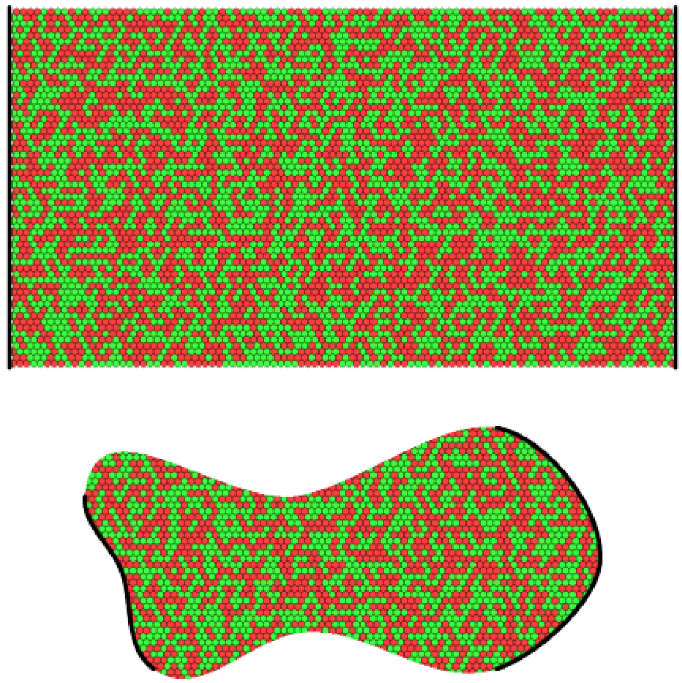


FIGURE 9. Percolation on two different shapes.[Kem17]

Some theoretical uses of the Schramm-Loewnwer evolutions is the explanation of dark matter halos and the evolution of the universe through the dark matter lens i.e evolution of dark matter halos. The motivation behind this comes from the Ising model which evolves in a very similar manner. Direct correlation between SLEs and dark matter halos would be crucial to answering questions in cosmology and about universe. However, there currently are applications to Quantum Field Theory and two-dimensional, incompressible Navier-Stokes.

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