# Longest Increasing Subsequences

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Euler Circle

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# What We will be covering Today

Here is a flow of what we will be covering today:

- What longest increasing subsequences are?
- Using Robinson-Schensted Algorithm to reach the limit shape
- Limit shapes and Plancherel random partitions
- The Limit Shape Theorem and Related theorems
- Final Result: Vershik, Kerov, Logan, and Shepp Limit Shape
- Additional Application of Limit Shape Theorem

# What Are Longest Increasing Subsequences

Let us take $S_n$ to denote a group of permutations of the order $n$. If $\sigma \in S_n$ is said to be a permutation then we can represent **sub-sequence** of $\sigma$ is a sequence $(\sigma(i_1), \sigma(i_2), ..., \sigma(i_k)$ where $1 \leq i_1 < i_2 < ... < i_k \leq n$. This sub-sequence could be either increasing, decreasing or monotone. $L(\sigma)$ can be defined as the maximum length of the increasing sub-sequence of $\sigma$

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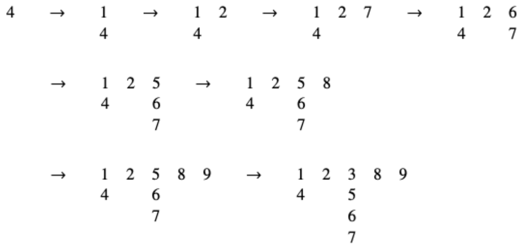
## Example 1

$\sigma = (1,4,5,7,9,0,8)$. For this example we can say that $L(\sigma) = 5$ since the longest increasing sub-sequence is of length 5. These longest increasing sub-sequences in relation to the example could be (1,4,5,7,9) and (1,4,5,7,8) both of which are increasing sub-sequences of length 5.

# The Robinson-Schensted Algorithm

It is a recursive application of the patience sorting algorithm. If we consider an example for patience sorting, say a permutation (4,1,2,7,6,5,8,9,3). When we run the patience sorting algorithm on this example we obtain the following figure.



# The Robinson-Schensted Algorithm



For this algorithm, instead of pushing the numbers down every time we want to add a new number, we bump them and re-apply patience sorting for that bumped number and so on. (Example from before) permutation (4,1,2,7,6,5,8,9,3). Here is a young diagram for the previous example but put through the algorithm:

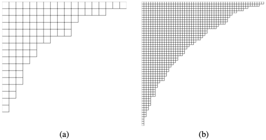
$$4 \rightarrow \begin{array}{l} 1 \\ 4 \end{array} \rightarrow \begin{array}{ll} 1 & 2 \\ 4 & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 7 \\ 4 & & \end{array} \rightarrow \begin{array}{lll} 1 & 2 & 6 \\ 4 & 7 & \end{array}$$

$$\rightarrow \begin{array}{lll} 1 & 2 & 5 \\ 4 & 6 & \\ 7 & & \end{array} \rightarrow \begin{array}{llll} 1 & 2 & 5 & 8 \\ 4 & 6 & & \\ 7 & & & \end{array}$$

$$\rightarrow \begin{array}{lllll} 1 & 2 & 5 & 8 & 9 \\ 4 & 6 & & & \\ 7 & & & & \end{array} \rightarrow \begin{array}{lllll} 1 & 2 & 3 & 8 & 9 \\ 4 & 5 & & & \\ 6 & & & & \\ 7 & & & & \end{array}$$

# Limit Shape of Plancherel-Random Partitions

Plancherel Measure of Order is the probability measure on the set of integer partitions of $n$ that assigns measure $d_\lambda^2/n!$ to any partition $\lambda$. When we apply it for $n \longrightarrow \infty$ . We can generate a Plancherel-random partition $\lambda^{(n)}$ of size $n$.
The Robinson–Schensted algorithm implies that $L(\sigma_n)$ is equal in distribution to the length $\lambda_n$ of the first row of a Plancherel-random partition $\lambda_n$ of order $n$.




Here the measures of order are: (a) $n = 100$ and (b) $n = 1000$
As can be observed, larger the value of $n$ gets, we obtain a smoother more continuous curve shape.

# The Limit Shape Theorem



## The Limit Shape Theorem

As $n \longrightarrow \infty$, the random function $\psi_n$ converges in probability in the norm $|| \cdot ||_\infty$ to the limiting shape $\Omega$ defined in (Theorem 1.1). That is, for all $\in \geq 0$ we have:

$$\mathbb{P}(sup_{u \in \mathcal{R}} |\psi_n(u) - \Omega(u)| > \in) \xrightarrow{n \longrightarrow \infty} 0$$

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Where Theorem 1.1 is: *As $n \longrightarrow \infty$, the random function $\psi_n$ converges in the metric $d_Q$ to the limiting shape in a particular probability which is given by:*

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \sin^{-1}(\frac{u}{2^{1/2}}) + \sqrt{2 - u^2}) & \text{if} \quad |u| \leq \sqrt{2} \\ |u| & \text{if} \quad |u| > \sqrt{2} \end{cases}$$

*That is, for all $\in > 0$ we can state that:*

$$\mathbb{P}(||\psi_n - \Omega||_Q > \in) \xrightarrow{n \longrightarrow \infty} 0$$

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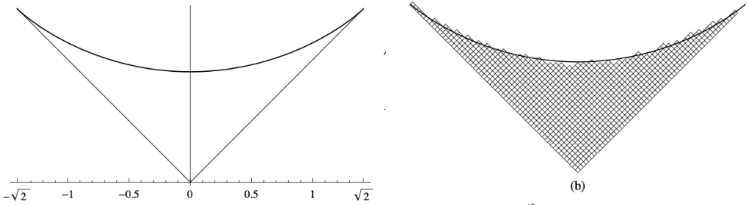
# Additional lemma

Let $A$, $L > 0$, and let $Lip(L, A)$ denote the space of all functions $f : R \longrightarrow R$ which are Lipschitz with constant $L$ and are supported on the interval $[A, A]$. For any $f \in Lip(L, A)$ we have

$$||f||_{\infty} \leq CQ(f)^{1/4}$$

where $C > 0$ is some constant that depends on A and L. This lemma is based on **fractional calculus**
As a consequence, we can prove the celebrated 1977 theorem of Vershik, Kerov, Logan, and Shepp. Which leads to the Limit Shape Theorem.

# Vershik, Kerov, Logan, and Shepp Limit Shape




Left: The Logan–Shepp–Vershik–Kerov limit shape Ω
Right: The limit shape superposed for comparison (after correct scaling) on a simulated Plancherel-random Young diagram of order n = 1000.
This is a remarkable feat of the Limit Shape Theorem.

# Applications

## Ulam-Hammersley problem

Let $S_{k;n}$ be a $k$-multiset permutation of size $n$ taken uniformly among the $\left( \begin{array}{c} Kn \\ kkkk...k \end{array} \right)$ possibilities. In the case k $=$ 1 the word $S_{1;n}$ is just a uniform permutation and estimating $L < (S_{1;n})$ is known as the Hammersley or Ulam-Hammersley problem.

*Idea of studying the statistical distribution of the maximal monotone subsequence length in a random permutation*

# Conclusion

Here's what we discussed:

- Longest Increasing Subsequences
- Patience Sorting
- The Robinson-Schensted Algorithm
- Limit Shape and Plancherel Measure of Order
- The Limit Shape Theorem
- Related Theorems
- Vershik, Kerov, Logan, and Shepp Limit Shape
- Application: Ulam-Hammersley problem

# Read More



**"Mathematics is a journey not a destination"**

- The Surprising Mathematics of Longest Increasing Subsequences by Dan Romik [Rom15]
- Limit shapes, real and imaginary by Andrei Okounkov [Oko]

Andrei Okounkov.
Limit shapes, real and imaginary.

Dan Romik.
*The Surprising Mathematics of Longest Increasing Subsequences.*
Cambridge University Press, 2015.