Longest Increasing Sub-sequences

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Abstract

In this mathematical paper, we explore the fascinating connection between the limit shape theorem and the Robinson-Schensted algorithm. The limit shape theorem is a fundamental result in the field of probability theory, which provides insight into the behavior of large random objects. On the other hand, the Robinson-Schensted algorithm is a powerful tool that relates permutations to pairs of standard Young tableaux. This paper works on these two results which hold the potential to facilitate advancements in practical applications, such as analyzing large-scale data sets and understanding the behavior of complex systems.

1 Introduction

The study of longest increasing sub-sequences and Limit Shape theorem as explored in this paper follows a certain unique order/flow which builds our knowledge base ultimately leading the reader to the final result.

1.1 Longest Increasing Sub-sequences

We begin by discussing what Longest increasing sub-sequences are. We discuss how the sub-sequences are subsets of a group of permutations and how they can be increasing, decreasing or monotone. This section also focuses on a new measure namely $L(\sigma)$ which is a statistic measure of the length of the increasing sub-sequence.

1.1.1 Young Diagram and Young Tableaux

In this sub-section we discuss what young diagrams and young tableaux are. We also discuss what partitions are and how we use partitions to describe a Young diagram or vice-versa.

1.2 The Robinson-Schensted Algorithm

1.2.1 Patience Sorting

In this sub-section we learn of a method of sorting our sub-sequences by doing a stack arrangement otherwise known as patience sorting. This method is what further brings about the Robinson-Schensted Algorithm

1.2.2 The Robinson Schensted Algorithm

In this sub-section, the prime agenda is discussion of the algorithm itself. We apply patience sorting several times, similar to how addition done repetitively is multiplication. We also consider an analogy of bumper cars to understand this algorithm better.

1.2.3 Robinson-Schensted Algorithm with Young Tableau

Here we discuss how the Robinson-Schensted Algorithm is nothing but a mapping of the permutation σ to 3 variables which are denoted by the variables (λ, P, Q).

1.3 Plancherel Measure of Order

1.3.1 Plancherel measure

In this sub-section we understand what Plancherel measure of order really is, and how it relates to partitions and the previously discussed variable $\lambda^{(n)}$.

1.3.2 Limit Shape of Plancherel-random Partitions

In this sub-section the prime agenda is to observe a Limit-shape phenomenon in the way the Plancherel-random partition approaches a limiting curve(shape) as we make the order $n \longrightarrow \infty$.

1.4 The Limit Shape Theorem

1.4.1 Background Theorems and Lemmas for references

In this sub-section we take a note of various conditions, variables and equations that will be of much help in the further sections to come.

1.5 Non-Standard Topology Method: Limit Shape Theorem

1.5.1 Theorem 8.1

We discuss a theorem which gives use the equation for the limit shape Ω and we discuss its proof in relation to hook co-ordinates.

1.5.2 Lemma 8.2

This lemma helps us bring the convergence from the previous theorem into a uniform norm by the help of fractional calculus.

1.5.3 Fractional Calculus

We discuss several aspects of Fractional calculus to provide a background for readers who are not familiar with those aspects.

1.5.4 Proof of Lemma 8.2

After having discussed the aspects of fractional calculus we finally come to the main aspect of proving the required lemma.

1.5.5 The Limit Shape Theorem

We finally come to the main agenda which is the Limit shape theorem. We discuss the proof of the limit shape theorem as well.

1.6 1977 Theorem of Vershik, Kerov, Logan, and Shepp

In this section we discuss how the limit shape theorem is a mathematical representation of the above theorem as well as how the 1977 theorem geometrically represents the outcome of the limit shape theorem.

1.7 Application of Limit Shape Theorem: Ulam-Hammersley Problem

Last but not least is a concluding application which shows how we have used everything discussed in the paper to find out the value of $L(\sigma)$ which is what we started with.

2 Acknowledgements

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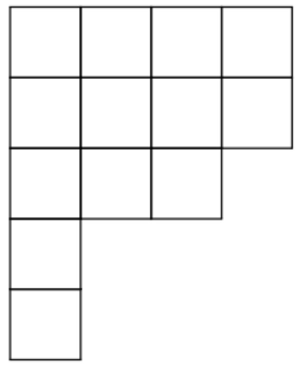
3 Longest Increasing Sub-sequence

The Limit shape theorem has many applications, however in order to fully understand the theorem itself it is imperative to understand several other theorems and concepts. We recall here some definitions, conventions and concepts which will be used throughout this paper.

Let us take S_n to denote a group of permutations of the order n. If $\sigma \in S_n$ is said to be a permutation then we can represent **sub-sequence** of σ is a sequence $(\sigma(i_1), \sigma(i_2), ..., \sigma(i_k)$ where $1 \leq i_1 < i_2 < ... < i_k \leq n$. This sub-sequence could be either increasing, decreasing or monotone. A sub-sequence is called **increasing** provided that $\sigma(i_1) < \sigma(i_2) < ... < \sigma(i_k)$, it is **decreasing** sub-sequence if $\sigma(i_1) > \sigma(i_2) > ... > \sigma(i_k)$, and is **monotone sub-sequence** if the sub-sequence is either increasing or decreasing.
Let us now define the symbol $L(\sigma)$ which will be used frequently throughout this paper. $L(\sigma)$ can be defined as the maximum length of the increasing sub-sequence of σ. Let us take an example, σ = (1,4,5,7,9,0,8). For this example we can say that $L(\sigma) = 5$ since the longest increasing sub-sequence is of length 5. These longest increasing sub-sequences in relation to the example could be (1,4,5,7,9) and (1,4,5,7,8) both of which are increasing sub-sequences of length 5.

3.1 Young Diagram and Young Tableaux

In order to understand a young diagram we need to talk about **partitions**. For $n \in N$ a partition of n can be explained as a method to represent n as the sum of positive integers without consideration of their order. For any particular partition of n say λ we can denote this by $\lambda \vdash n$. We can take an example where n is 5. Some possible partitions include (1+4) and (2+3). Taking this example in consideration, parts are 1,4 or 2,3. Parts in other words are the integers making up the partition. We also call n the **size** of λ. For a partition $\lambda \vdash n$, we can define the Young diagram of λ to be a graphical diagram representing the partition λ as a two-dimensional array of boxes, or cells, where the jth row (counting from the top down) has λ_j cells, and the rows are left-justified. For example the following Young Diagram corresponds to the partition(4,4,3,1,1).



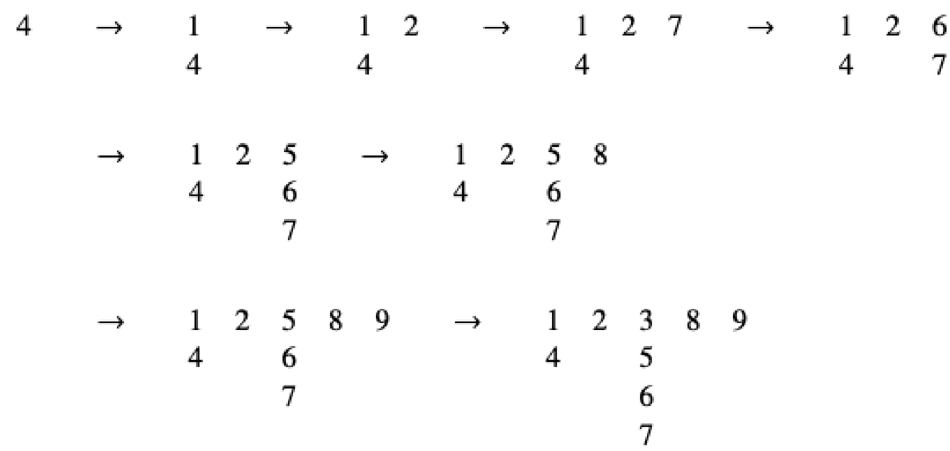
The above definitions will come handy in the next section on The Robinson-Schensted Algorithm.

4 The Robinson-Schensted Algorithm

The Robinson-Schensted Algorithm is a recursive application of the patience sorting algorithm(as will be discussed below). This is a very important and remarkable result which helps us in analyzing the probabilistic behavior of the permutation statistic $L(\sigma_n)$, where σ_n is chosen uniformly at random in S_n.

4.1 Patience Sorting

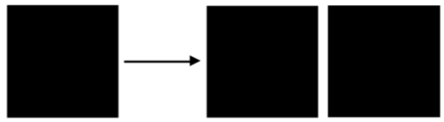
is an algorithm which suggests its application to sorting a deck of cards. It consists of sequentially going through the permutation values in order, and piling them in a linear array of "stacks," according to the following rules: 1. Each new value x will be placed at the top of an existing stack, or will form a new stack by itself positioned to the right of all existing stacks. 2. The stack on which each new value x is placed is the leftmost stack from among those whose current top number is bigger than x. If there are no such stacks, x forms a new stack. If we consider an example for patience sorting, say a permutation (4,1,2,7,6,5,8,9,3). When we run the patience sorting algorithm on this example we obtain the following figure. We can notice how adding each new number results in the stack being pushed down.



4.2 The Robinson Schensted Algorithm.

In order to compute the value of $L(\sigma)$, which was our primary goal from the start, we only require the number of stacks, which is the first row of the array. This is because they are the sole deciding factors to decide where to place each new number. We can thus package the other information which is of no use to us and shifting them. We shift them by taking the numbers pushed, or "bumped," down to the second row and using them as input for an additional patience sorting algorithm, and so on recursively for the third row, fourth row, and so on.

- What happens when a new number is inserted. The number settles down into the first row at the top of one of the stacks (now more appropriately considered simply as columns of bumper cars, more on this below)
- As it does so, it either makes its own new column to the right of the existing ones, in which case this the last step
- Or, if it settles down at the top of an existing column, instead of pushing down as we did before, we can simply "bump" the previous top entry from that column down to the second row. Imagine 3 bumper cars, 2 in a line and one separately(represented as dots) when the 3rd car bumps into the line, the last car gets removed.
- The number which is bumped down, now has a new playing field, namely row 2(or the next row) where it follows the same steps again, this time ignoring the previous row(s).



- It now searches for a column to settle down in, following similar rules to the standard patience sorting algorithm:
  - Either it settles in an empty space to the right of all other second-row numbers if it is bigger than all the numbers currently in the second row
  - Otherwise it settles down in the leftmost column having a second-row entry bigger than itself.

These cycles of inserting new values and "bumping" down previous values is called an **insertion step**.

This is the Robinson-Schensted algorithm applied to the permutation (4,1,2,7,6,5,8,9,3)

```
4  →  1  →  1 2  →  1 2 7  →  1 2 6
      4     4       4         4 7

   →  1 2 5  →  1 2 5 8
      4 6       4 6
      7         7

   →  1 2 5 8 9  →  1 2 3 8 9
      4 6           4 5
      7             6
                    7
```

The **Robinson-Schensted Algorithm** in a summary can be defined as the procedure which applies patience sorting algorithm recursively to the rows below the main/first row. Involving young diagrams into the picture, we can add the final leg of the algorithm.

4.3 Robinson-Schensted Algorithm with Young Tableau

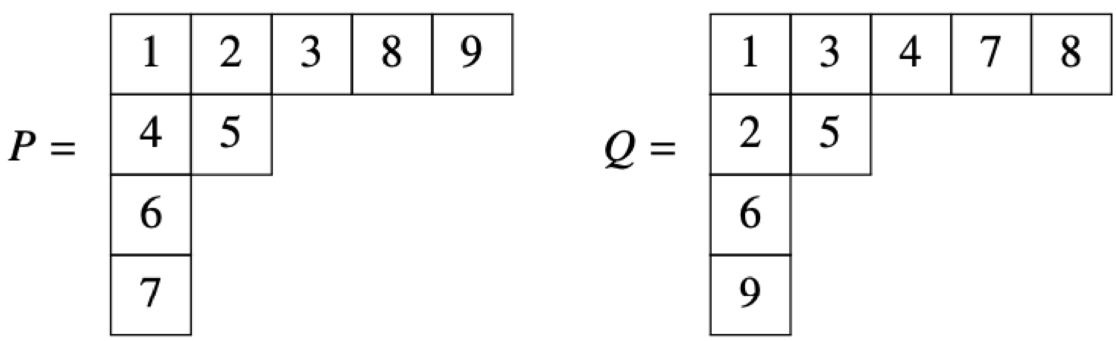
Let us assume 3 variables for the same.

1. Young Tableau λ which is a graphical diagram representing the partition λ as a two-dimensional array of boxes, or cells, where the jth row (counting from the top down) has λj cells, and the rows are left alligned.

2. P is the **insertion tableau**. It is the Young Tableau of shape λ used to compute permutation $\sigma \in S_n$ and partition $\lambda \vdash n$.

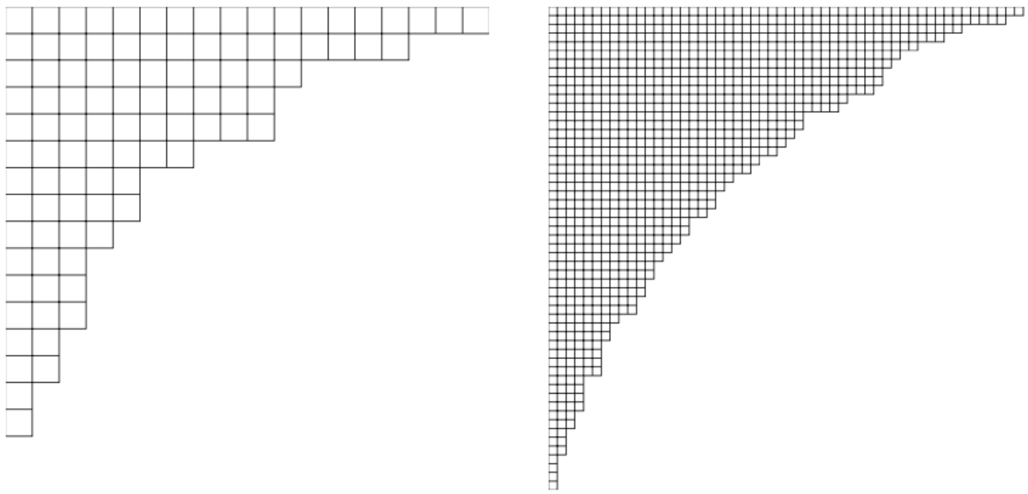
3. Q is also a young tableau of shape λ, by recording in each cell of the Young diagram λ the number k if that cell first became occupied in P during the kth insertion step during the execution of the algorithm.

   The Robinson–Schensted algorithm is defined as the mapping taking a permutation σ to the triple (λ, P, Q).



5 Plancherel Measure of Order

The Robinson-Schensted algorithm is pretty useful, but one of it's results which are of most importance to us in context of the Limit Shape Theorem is the Plancherel Measure. The algorithm allows us to obtain an alternative way of looking at permutations, where $L(\sigma)$ is the length of the first row of the Robinson-Schensted shape λ. When we apply the Robinson Schensted algorithm to a random permutation σ_n we obtain a random Young diagram $\lambda(n)$.



(a) (b)

Figure 1. fig 1

5.1 Plancherel measure

(of order n) can be stated as the probability measure on the set of integer partitions of n that assigns measure $d_\lambda^2/n!$ to any partition λ. We get this by probabilistically analysing how the random $\lambda^{(n)}$ behaves. We get that for any $\lambda \vdash n$, the probability that $\lambda^{(n)} = \lambda$ is $1/n!$ times number or permutations $\sigma \in S_n$ comes out as:

$$\mathbb{P}(\lambda^{(n)} = \lambda) = d_\lambda^2/n!$$

History Fact This measure of order historically originated in the works of Michel Plancherel, the Swiss mathematician, in his work on the representation theory, in the early 20th century.

5.2 Limit Shape of Plancherel-random Partitions

We will now do an analysis of the asymptotic behaviour of random partitions of n which will be chosen using the above, i.e. Plancherel measure as $n \longrightarrow \infty$. We can generate a Plancherel-random partition $\lambda^{(n)}$ of size n, by applying the Robinson-Schensted algorithm to a uniformly random permutation $\sigma_n \in S$. We can see the results for the partition of different orders as generated by a computer. The Robinson–Schensted algorithm implies that $L(\sigma_n)$ is equal in distribution to the length λ_n of the first row of a Plancherel- random partition λ_n of order n. Here(in fig 1) the measures of order are: (a) $n = 100$ and (b) $n = 1000$

As can be observed, larger the value of n gets, we obtain a smoother more continuous curve shape. This fascinating observation is not specific to the current example, it is in fact a general result. This result, that we obtain is actually an instance of the **limit shape** phenomenon(we will discuss the limit shape theorem later on in the paper). In a nutshell the limit shape phenomenon is how some geometric object which is related to some random distinct objects that converge to an almost continuous(or limiting continuous) shape as one of its parameters(in this case the order) tends to infinity.

Let $\mathcal{P}(n)$ be the set of partitions of n. By using the function $p(n)$ which is also called the **partition function** we can arrive at the following probability measure on the set $\mathcal{P}(n)$.

$$p(n) = |\mathcal{P}(n)|$$

Fun fact: The function $p(n)$ us an important function of number theory. Its first few values are 1,2,3,5,7,11,15...

This concept or rather the instance of the Limit shape phenomenon brings us to the Limit Shape Theorem itself.

6 The Limit Shape Theorem

In order to obtain the limit shape theorem we will first have to consider a few theorems and lemmas in order to reach the final result. Before beginning with the above, we must first define a few symbols and values:

6.1 Background Theorems and Lemmas for reference

For each $n \geq 1$ we let $\lambda^{(n)}$ denote a random partition of order n chosen with respect to the Plancherel measure. Also let $\phi_n(x)$ be a function in the function space $\mathcal{F}$ with respect to:

$$\phi_\lambda(x) = n^{-1/2}\lambda'_{[n^1/2x]+1}, (x0).$$

Where we can define $\mathcal{F}$ to be the space of functions $f : [0, \infty) \longrightarrow [0, \infty)$ such that:

1. f is non-increasing;
2. $\int_{-0}^{\infty} f(x)dx = 1$;
3. f has compact support, that is, sup $(x \geq 0 : f(x) > 0) < \infty$.

Let us also define a element ψ_n in a new space of functions $\mathcal{G}$ that describe the limiting shapes of Young diagrams in a new co-ordinate system which is rotated, refer to hook coordinates in references. As per hook co-ordinates let

- $\phi_n = f$
- $\psi_n = g$

For $v = g(u) \Longleftrightarrow (v - u)/2^{1/2} = f((v + u)/2^{1/2})$

Proposition: *For any Lipschitz function $h : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support, we have that $Q(h) \geq 0$, with equality if and only if $h \equiv 0$.*

All of the above is but a mathematical and more defined way of describing something which we have already been using. The function ψ_n when graphed is the young diagram resulting from Plancherel-random partitions as discussed below. The only difference is that this diagram is to be rotated in accordance with hook coordinates.

7 Non-Standard Topology Method: Limit Shape Theorem

Using Lipschitz function, $\mathcal{G}$ and Q we can denote:

$$||h||_Q = Q(h)^{1/2}$$

Furthermore the proposition stated above enables us to define the induced metric d_Q on $\mathcal{G}$ as follows:

$$d_Q(g_1, g_2) = ||g_1 - g_2||_Q = Q(g_1 - g_2)^{1/2}$$

$(g_1, g_2 \in \mathcal{G})$ This leads us to the very first theorem in relation to the Limit Shape Theorem.

Theorem 7.1. *As $n \longrightarrow \infty$, the random function ψ_n converges in the metric d_Q to the limiting shape in a particular probability which is given by:*

$$\Omega(u) = \begin{cases} \frac{2}{\pi}(u \sin^{-1}(\frac{u}{2^{1/2}}) + \sqrt{2-u^2}) & if \quad |u| \leq \sqrt{2} \\ \\ |u| & if \quad |u| > \sqrt{2} \end{cases}$$

That is, for all $\in > 0$ we can state that:

$$\mathbb{P}(||\psi_n - \Omega||_Q > \in) \xrightarrow{n \longrightarrow \infty} 0$$

Proof. For each partition $\lambda \in \mathcal{P}(n)$, denote by g_λ the element of $\mathcal{G}$ corresponding to the function $\psi_\lambda \in \mathcal{F}$, and denote $h_\lambda(u) = g_\lambda(u) - |u|$. Denote by $\mathcal{M}_n$ the set of partitions $\lambda \in \mathcal{P}(n)$ for which $||g_\lambda - \Omega||_Q > \epsilon$. By the following,

$$J(h) \geq J(h_o) + Q(h - h_o)$$

(Which is a result from Logan, B. F., and Shepp, L. A. 1977. A variational problem for random Young tableaux. Adv. Math., 26, 206–222.) For each $\lambda \in \mathcal{M}_n$ we have:

$$J(h_\lambda) \geq J(h_o) + Q(h_\lambda - h_o) > -1 + e^2$$

By using various results from transformation to hook co-ordinates we get the following:

$$\mathbb{P}(\lambda^{(n)} = \lambda) \leq exp(-\epsilon^2 n + O(\sqrt{n} logn))$$

where the term O can be said to be uniform for/over all partitions of n. We can also note a useful result $|\mathcal{M}_n| \leq |\mathcal{P}(n)| \leq e^{C\sqrt{n}}$. Using which we can state that:

$$\mathbb{P}(\lambda^{(n)} \in \mathcal{M}_n) = \sum_{\lambda \in \mathcal{M}_n} \mathbb{P}(\lambda^{(n)} = \lambda) \leq C exp(-\epsilon^2 n + C\sqrt{n} + O(logn\sqrt{n})$$

which as is evident will converge to 0 as we keep increasing n or in other words as $n \longrightarrow \infty$ ∎

The above theorem is a useful result, however it has an issue. It helps us prove that convergence to the limit shape theorem in an exotic topology in unclear. This makes it difficult for us to apply this theorem to our work. In order to correct this difficulty we will use the following lemma, thus getting a convergence in the supremum norm.

Lemma 7.2. *Let $A, L > 0$ and let Lip(L, A) denote the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are Lipschitz with constant L and are supported on the interval $[-A, A]$. For any $f \in$ Lip(L, A) we have*

$$||f||_\infty \leq CQ(f)^{1/4}$$

Where $C > 0$ is some constant that depends on A and L.

We can prove the above lemma using some certain results or facts from **fractional calculus**. By using results of the Fourier analysis and Plancherel theorem we can take the quantity $Q(f)^{1/2}$ as proportional to the supremum norm of a function g which can be defined as $g : \mathbb{R} \longrightarrow \mathbb{C}$ and also whose Fourier transformation can satisfy :$|\hat{g}(s)| = |s|^{1/2}|\hat{f}(s)|$. This function is the half-derivative, and thus it makes sense to use **fractional calculus** to prove the theorem. We get the function g as follows:

$$g(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{f(x) - f(x-t)}{t^{3/2}} dt = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^x \frac{f(x) - f(y)}{(x-y)^{3/2}} dy$$

By using the previously stated fact that f is defined on [-A,A] and Lipschitz properties we can conclude that our integral(function g) converges definitively for any x. As a result we can represent g(x) as the following function:

$$g(x) = \begin{cases} 0 & \text{if} \quad x \leq -A \\ \frac{f(x)}{\sqrt{\pi(x+a)}} + \frac{1}{2\sqrt{\pi}} \int_{-A}^x \frac{f(x)-f(y)}{(x-y)^{3/2}} dy & \text{if} \quad -A < x < A \\ -\frac{1}{2\sqrt{\pi}} \int_{-A}^A \frac{f(y)}{(x-y)^{3/2}} dy & \text{if} \quad x \geq A \end{cases}$$

Before we start with the core part of the proof it is imperative that we discuss some ideals of **fractional calculus** for clarity purposes.

Note for readers: feel free to skip the below sub-section if you feel you are aware of fractional calculus.

7.1 Fractional Calculus

While fractional calculus exists as an entire study in itself there is no unique way in which we may define the operation, meaning of the output or the relation between fractional integral and fractional derivative operator. It is entirely dependent on what the functional space and type of operator used is.

One method/approach is the **Riemann-Liouville integral** which is defined as below:

$$(I_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{-1+\alpha} dt$$

In this integral the symbol $\Gamma(\bullet)$ denotes the Euler Gamma function. If $Re(\alpha) > 0$ then this operator (the integral) is said to be defined. In certain conditions we can prove that the inverse operation for the above is nothing but the **Marchaud fractional derivative operator** D_α, which can be defined as below:

$$(D_\alpha g)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt$$

Reference: [Mil]

7.2 Proof

Proof. From our representation, we can come to the conclusion that our function g is a bounded function and that for a value of x which is categorised as large and positive will satisfy a bound $|g(x) \leq Cx^{-3/2}$ where C is a positive constant. The function g has a Fourier transform (g is in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$) which we can compute as follows:

$$\hat{g}(s) = \int_{-\infty}^{\infty} g(x) s^{-isx} dx = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{t^{3/2}} \int_{-\infty}^{\infty} (f(x) - f(x-t)) e^{-isx} dx dt$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{t^{3/2}} (\hat{f}(s) - \hat{f}(s)) e^{-ist} dt$$

$$= \frac{1}{2\sqrt{\pi}} |s|^{1/2} \hat{f}(s) \int_0^\infty \frac{1 - e^{-isgn(s)u}}{u^{3/2}} du = |s|^{1/2} e^{-\pi isign(s)/4} \hat{f}(s)$$

Where we use the integral evaluation:

$$\int_0^\infty \frac{1 - e^{iu}}{u^{3/2}} du = 2s^{-\pi i/4} \sqrt{\pi}$$

This gets us the exact property that we needed because with respect to the above we get $||g||_2^2 = ||\hat{g}||_2^2/2\pi = (2/\pi)Q(f)$.

Now we will compare the supremum function $||f||_\infty$ with the above(in rearranged format) that is $Q(f) = (\pi/2)||g||_2^2$. However we cannot jump into this directly. In order to begin we need to know how we can find f from g.

This step can be done by operating a fractional *integration* of 1/2 order, as is follows:

$$g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} \frac{g(t)}{\sqrt{x-t}} dt = \begin{cases} 0 & \text{if} \quad x < -A \\ -\frac{1}{\sqrt{\pi}} \int_{-A}^{x} \frac{g(t)}{\sqrt{x-t}} dt & \text{if} \quad x \geq -A \end{cases}$$

The Fourier transformation of h can be defined as below:

$$\hat{h}(s) = \int_{-A}^{\infty} h(x) e^{-isx} dx = \frac{1}{\sqrt{\pi}} \int_{-A}^{\infty} e^{-isx} \int_{-A}^{x} \frac{g(t)}{\sqrt{x-t}} dt dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-A}^{\infty} g(t) \int_{t}^{\infty} \frac{e^{-isx}}{\sqrt{x-t}} dx dt = \frac{1}{\sqrt{\pi}} \int_{-A}^{\infty} g(t) e^{-ist} dt \int_{0}^{\infty} \frac{e^{-isu}}{\sqrt{u}} du$$

$$= \frac{1}{\sqrt{\pi}} |s|^{-1/2} \hat{g}(s) \int_{0}^{\infty} \frac{e^{-isgn(S)x}}{\sqrt{x}} dx = |s|^{-1/2} e^{\pi i sgn(s)/4} \hat{g}(s) = \hat{f}(s)$$

Over here $s \neq 0$, so we use another integral which is in relation/rather equal to the previously used integral evaluation:

$$\int_{0}^{\infty} \frac{1 - e^{iu}}{u^{3/2}} du = 2e^{-\pi i/4} \sqrt{\pi}$$

and is actually:

$$\int_{0}^{\infty} \frac{e^{iu}}{u^{1/2}} du = e^{\pi i/4} \sqrt{\pi}$$

By the above we get that the function $h = f$ for almost all values. We hence can conclude that for this particular fractional calculus application the half-derivative(the function g(x)) is indeed the inverse of the half-integral(the function g(x)). This is not concrete however, and only a computation because we did not define the function h to belong to the same norm as g that is $L^1(\mathbb{R})$ and thus the Fourier transform that we conducted may or may not be justified. Irrespective of the uncertainties we can conclude that $h = f$.

Let us take the supremum norm of f L^1 again and bound it in terms of $||g||_2 = \sqrt{2/\pi}Q(f)^{1/2}$:

$$||f||_1 = \int_{-A}^{A} |f(x)|dx \leq \frac{1}{\sqrt{\pi}} \int_{-A}^{A} \int_{-A}^{x} \frac{|g(t)|}{\sqrt{x-t}} dtdx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-A}^{A} |g(t)| \left(\int_{t}^{x} \frac{1}{\sqrt{x-t}} dx \right) dt = \frac{1}{\sqrt{\pi}} \int_{-A}^{A} |g(t)| 2\sqrt{A-t} dt$$

$$\leq \frac{2\sqrt{2A}}{\sqrt{\pi}} \int_{-A}^{A} |g(t)|dt \leq \frac{4A}{\sqrt{\pi}} \left(\int_{-A}^{A} g(t)^2 dt \right)^{1/2}$$

$$\leq \frac{4A}{\sqrt{\pi}} ||g||_2 = \frac{4\sqrt{2}}{\pi} AQ(f)^{1/2}$$

We can now convert the above to a more uniform norm of f. Let us define a variable x_o such that $x_o \in [-A, A]$ so that we have $|f(x_o)| = ||f||_\infty$. Using the Lipschitz property we can get the final result that is:

$$||f||_1 = \int_{-A}^{A} |f(x)|dx = \int_{-\infty}^{\infty} |f(x)|dx$$

$$\geq \int_{-\infty}^{\infty} max(0, |f(x_o)| - L|x - x_o|)dx = \frac{|f(x_o)|^2}{L} = \frac{||f||_\infty^2}{L}$$

■

As a consequence of the above we finally prove the required Lemma.

Theorem 7.3. *(Limit shape theorem for Plancherel-random partitions): As $n \longrightarrow \infty$, the random function ψ_n converges in probability in the norm $|| \cdot ||_\infty$ to the limiting shape Ω defined in (Theorem 1). That is, for all $\in \geq \theta$ we have:*

$$\mathbb{P}(sup_{u \in \mathcal{R}} |\psi_n(u) - \Omega(u)| > \in) \xrightarrow{n \longrightarrow \infty} 0$$

Proof. By using the following lemma: Lemma: For any $\alpha > e$ we get:

$$\mathbb{P}(L(\sigma_n) > \alpha\sqrt{n}) \leq Ce^{-c\sqrt{n}}$$

as well as the Robinson Schensted Algorithm, we can see that as the order $n \longrightarrow \infty$, our plancherel random partition $\lambda^{(n)}$ has a first row which is of length $< 3\sqrt{n}$ and will equally apply to the column length as well. Thus by using the lemmas and theorems as described above we get a convergence which exists for the uniform norm. ■

8 1977 Theorem of Vershik, Kerov, Logan, and Shepp.

The Logan-Shepp-Verkshik-Kerov theorem is in relation to the Limit Shape theorem and the various other theorems and lemmas that have been discussed in this paper so far.

As discussed in Theorem 1 we get the graph for the function $\Omega(u)$. We also get the functional graph or rather the young diagram, obtained from the Robinson-Schensted Algorithm and Plancherel Measure of Random-Partitions of order n. This graph is then transformed using the logic of hook co-ordinates. This gives us the function $\psi_n(u)$. As we can observe using the below diagram, as the order $n \longrightarrow \infty$ our function graph(b) that is $\psi_n(u)$ approaches the graph(a) of $\Omega(u)$. This graph (a) is the limit shape as defined by Logan, Shepp, Vershik and Kerov.

The Limit shape theorem itself states what we have stated above. As per the notation used we can see that for an order $n \longrightarrow \infty$ the probability that there be a difference(absolute magnitude only) greater than 0, between the values of function $\Omega(u)$ and $\psi_n(u)$, itself tends to 0. That is the graphs of both functions will tend to overlap(or converge).

This is the final result which can be concluded from all things discussed in this paper.

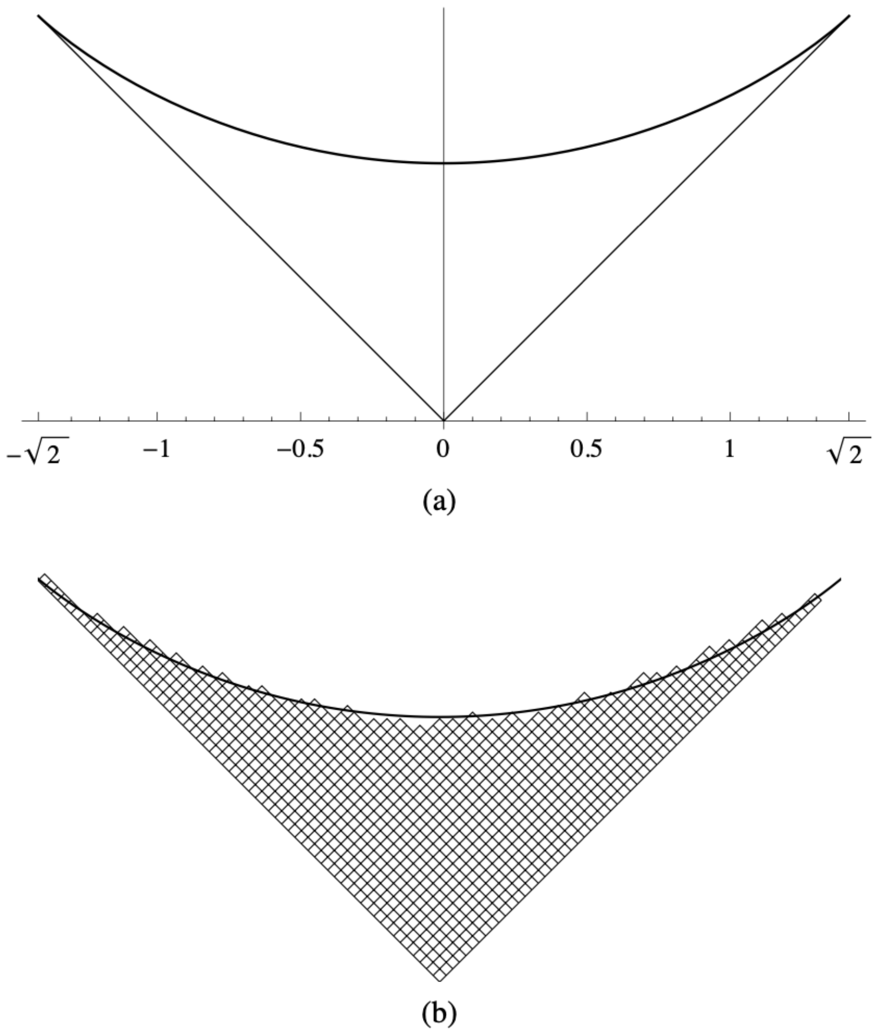


Figure 1.13 (a) The Logan–Shepp–Vershik–Kerov limit shape Ω. (b) The limit shape superposed for comparison (after correct scaling) on a simulated Plancherel-random Young diagram of order $n = 1000$.

9 Application of Limit Shape Theorem: Ulam-Hammersley Problem

The Ulam-Hammersley Problem can be described as below:

Let $S_{k;n}$ be a k-multiset permutation of size n taken uniformly among the $\begin{pmatrix} Kn \\ kkkk...k \end{pmatrix}$ possibilities. In the case k = 1 the word $S_{1;n}$ is just a uniform permutation and estimating $L < (S_{1;n})$ is known as the Hammersley or Ulam-Hammersley problem.

In much simpler words we can describe it as: *Idea of studying the statistical distribution of the maximal monotone sub-sequence length in a random permutation*

The Ulam-Hammersley problem discusses what we have tried solving since the beginning that is: what is the length of the longest increasing sub-sequences $L(\sigma)$. Which we can now answer with help from the Limit Shape theorem as: The Robinson–Schensted algorithm implies that $L(\sigma_n)$ is equal in distribution to the length λ_n of the first row of a Plancherel-random partition λ_n of order n. And we can now find that using the Limit Shape theorem.

10 Conclusion

In this paper we have taken a roller coaster ride starting from the basic theorems and concepts towards a final result which is so wholesome. We learnt about longest increasing sub sequences and the Robinson-Schensted Algorithm. Using those tools we arrived at Limit Shape of Plancherel-random partitions which gave way to the Limit Shape Theorem. Through the series of lemmas and theorems we finally arrived at the Limit Shape Theorem. This theorem was what helped us solve the Logan-Shepp-Verkshik-Kerov theorem and Ulam-Hammersley problem. Both of these problems led us right back to the start thus giving us the answers we looked for.

Citations/References

"Mathematics is a journey not a destination"

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Note: Images, except for bumper car diagram have been sourced from [Rom15]